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Bootstrap Method for Testing of Equality of Several Coefficients of Variation

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Abstract The coefficient of variation has been found to be very useful unit less measure of relative consistency of sample data in many areas such as chemical experiments, finance, insurance risk assessment, medical studies, etc., a chi-square test is used for testing the equality of several coefficients of variation in the literature. This chi-square test demonstrates only the statistical significance of coefficients of variation. In this paper, a bootstrap graphical method is developed as an alternative to the chi-square test to test the hypothesis on equality of several coefficients of variation. An example is given to demonstrate the advantage of bootstrap graphical procedure over the chi-square test from decision making point of view.
Keywords *Coefficient of Variation, Chi-Square Test, Bootstrap Method*

1. Introduction

Coefficient of variation is used in such problems where we want to compare the variability of two or more than two groups. The series for which the coefficient of variation is greater is said to be more variable or conversely less consistent, less uniform, less stable or less homogeneous. On the other hand, the series for which coefficient of variation is less is said to be less variable or more consistent, more uniform, more stable or more homogeneous. The coefficient of variation is independent of unit of measurement and has been found to be a very useful measuring of relative consistency of sample data in many situations. For example, the coefficient of variation is useful in measure risk assessment as a measure of the heterogeneity of insurance portfolios. Coefficient of variation is also used in comparing the characteristics such as tensile strengths, weights of materials, etc. in the processing type of industries.

Statistical inference based on data resampling has drawn a great deal of attention in recent years. The main goal is to understand a collection of ideas concerning the non-parametric estimation of bias, variance and more general measures of errors. The main idea about these resampling methods is not to assume much about the underlying population distribution and instead tries to get the information about the population from the data itself various types of resampling leads to various types of methods like the jackknife and the bootstrap. Bootstrap method (Efron, 1979) use the relationship between the sample and resamples drawn from the sample, to approximate the relationship between the population and samples drawn from it. With the bootstrap method, the basic sample is treated as the population and a Monte Carlo style procedure is conducted on it. This is done by randomly drawing a large number of resamples of size n from this original sample with replacement.

Both bootstrap and traditional parametric inference seek to achieve the same goal using limited information to estimate the sampling distribution of the chosen estimator $\hat{\theta}$. The estimate will be used to make inferences about a population parameter θ. The key difference between these inferential approaches is how they obtain this sampling distribution whereas traditional parametric inference utilizes a priori assumptions about the shape of distribution of $\hat{\theta}$. The non-parametric bootstrap is distribution free which means that it is not dependent on a particular class of distributions. With the bootstrap method, the entire sampling distribution of $\hat{\theta}$ is estimated by relying on the fact that the sampling distribution is a good estimate of the population distribution. In section 3, bootstrap method applied to testing of equality of several coefficients of variation is explained [1, 2].

2. Testing of Equality of Several Coefficients of Variation

Let $\{X_{ij},\ i=1,2,\ldots,k,\quad j=1,2,\ldots,n\}$ represent k independent random samples of size n and we assume that $X_{ij} \sim N(\mu_i, \sigma_i^2)$ for $i=1,2,\ldots,k$. Since the k samples are drawn from k normal populations with different means and different variances, the coefficient of variation $\gamma = \frac{\sigma}{\mu}$ is a useful characteristic to measure the relative variability in the k normal populations. Here, we are interested in testing the null hypothesis. $H_0 : \gamma_1 = \gamma_2 = \ldots = \gamma_k = \gamma$ (Unknown), where $\gamma_i = \frac{\sigma_i}{\mu_i}$ against the alternative hypothesis that at least two coefficients of variation are unequal.

Chi-square test is used for testing H_0 in the literature [3, 4]. This test demonstrates only the statistical significance of the coefficients of variation being compared. Chi-square test for testing H_0, Miller and Feltz (1997) suggested a test statistic and it is given by

$$\chi^2 = \frac{m\sum_{i=1}^{k}(c_i - \bar{c})^2}{(0.5+\bar{c}^2)\bar{c}^2} \sim \chi^2_{(k-1)} \text{ (Under } H_0) \tag{2.1}$$

Where $m = n-1,\ \bar{x}_i = \frac{\sum_{j=1}^{n} x_{ij}}{n},\quad s_i = \left(\frac{1}{n-1}\sum_{j=1}^{n}(x_{ij}-\bar{x}_i)^2\right)^{\frac{1}{2}}, c_i = \frac{s_i}{\bar{x}_i} \quad and \quad \bar{c} = \frac{1}{k}\sum_{i=1}^{k} c_i$

We reject the null hypothesis, H_0 if $\chi^2 \geq \chi^2_{k-1,\alpha}$

3. Bootstrap Graphical Method for Testing of Equality of Several Coefficients of Variation

Let $\{X_{ij}, i=1,2,\cdots k;\ j=1,2,\cdots n\}$ represent k available independent random samples of size n and the coefficient of variation of the ith sample is given by $c_i = \frac{s_i}{\bar{x}_i}$ for i=1, 2...k. Bootstrap graphical procedure for testing the equality of several coefficients of variation is given in the following steps.

1. Let Y_{ijb} be the b-th bootstrap sample of size n, drawn from ith available sample, where b=1, 2…B (=3000), i=1, 2…k and j=1, 2…n.
2. Compute $\bar{y}_{ib}$ and s_{ib}, the mean and standard deviation of b-th bootstrap sample form ith available sample and are given by $\bar{y}_{ib} = \frac{1}{n}\sum_{j=1}^{n} Y_{ijb}$ and $s_{ib} = \sqrt{\left(\frac{1}{n-1}\sum_{j=1}^{n}\left(Y_{ijb} - \bar{y}_{ib}\right)^2\right)}$.
3. Compute c_{ib}, be the coefficient of variation of b-th bootstrap sample from ith available sample and is given by $c_{ib} = \frac{s_{ib}}{\bar{y}_{ib}}$, i=1, 2…k and b=1, 2…B.
4. Compute $c_b = \frac{1}{k}\sum_{i=1}^{k} c_{ib}$, b=1, 2…B.
5. Obtain the sampling distribution of coefficient of variation using B-bootstrap estimates and compute the central decision line (CDL) as $\bar{c}^* = \frac{1}{B}\sum_{b=1}^{B} c_b$ and the standard error is given by $SE(c^*) = \sqrt{\frac{1}{B}\sum_{b=1}^{B}\left(c_b - \bar{c}^*\right)^2}$. The lower decision line (LDL) and the upper decision line (UDL) for the comparison of each of the c_i are given by

   $$LDL = \bar{c}^* - z_{\alpha/2} SE(c^*)$$
   $$UDL = \bar{c}^* + z_{\alpha/2} SE(c^*)$$

   Where z_α is the α -th upper cut off point of standard normal distribution.
6. Plot c_i against the decision lines. If any one of the points plotted lies outside the respective decision lines, H_0 is rejected at 5% level and conclude that the coefficients of variation are not homogenous.

The proposed method is very useful in handling of small samples of size less than 30. This method not only tests the significant difference among the coefficients of variation but also identify the source of heterogeneity of coefficients of variation.

Size of the proposed test is obtained using simulation of random samples from normal populations having with the equal coefficient of variations.
Let the populations,
$X_1 \sim N(2,1),\ X_2 \sim N(4,4),\ X_3 \sim N(6,9),\ X_4 \sim N(8,16)\ and\ X_5 \sim N(10,25)$ having with the same coefficients of variation. The proposed test procedure is performed 100 times to compare the k populations with respect to coefficients of variation using the different samples (equal in size) drawn

from the above five populations. The size of the test is defined as number of times the test procedure rejecting the null hypothesis of equality of coefficients of variations in 100 iterations.
That is,

$\alpha = \frac{\text{Number of times the null hypothesis is rejected}}{100}$. The following table presents the size of the test for comparing k-population coefficients of variation based on the samples of size n= 5, 10, 15, 20, 25 and 30.

Table 1: *Size of the Proposed Test*

k\n	5	10	15	20	25	30
3	0.15	0.14	0.14	0.12	0.10	0.10
4	0.19	0.18	0.15	0.14	0.12	0.11
5	0.22	0.22	0.20	0.17	0.15	0.12

Power of the test procedure is computed using simulating random samples from normal populations. Let the populations,

$X_1 \sim N(2,2),\ X_2 \sim N(4,2),\quad X_3 \sim N(6,9),\ X_4 \sim N(4,4)\, and\ X_5 \sim N(5,5)$, the populations are considered in such a way that these are having with the different coefficients of variation across the populations. The test procedure is performed 100 times by considering the different samples from the k-populations. Let β be the ype-II error and which is computed as

$\beta = \frac{\text{Number of times accepting H}_0}{100}$. Power of the test is given by $1-\beta$ and is computed for comparison of k-populations based on the samples of size n=5, 10, 15, 20, 25 and 30. The following table presents the power of the proposed test.

Table 2: *Power of the Test*

k\n	5	10	15	20	25	30
3	0.85	0.85	0.87	0.89	0.91	0.93
4	0.82	0.84	0.87	0.86	0.89	0.91
5	0.80	0.85	0.88	0.89	0.94	0.93

In the above tables k represents the number of populations compared and n is the size of the each sample drawn from the k-populations in testing of equality of coefficients of variation. From the above two tables, it is observed that the size of the test is decreasing and the power of the test is increasing as the sample size increases. The proposed test procedure is explained with a numerical example in the following section.

4. Numerical Example

Example 5.2 from the paper of Tsou (2009) is considered and this example describes the numbers of birth in 1978 on Monday, Thursday, and Saturday in the United Kingdom. We use the new procedure to test whether the coefficients of variation of the three different dates are the same [5]. Let c_1 represents the coefficient of variation of numbers of birth on Monday, c_2 represents the coefficient of variation of numbers of birth on Thursday and c_3 represents the coefficient of variation of numbers of birth on Saturday. For the given data c_1=0.0649, c_2=0.0580, c_3=0.0465, k=3 and n=52. We obtain χ^2 test statistic value is 5.5079 and the significant value at 5% level is $\chi^2_{2,0.05} = 5.9915$.

Since the test statistic value is less than the critical value, therefore we accept H_0 at 5% level.

By applying the bootstrap procedure explained in Section 3, the LDL, CDL and UDL are obtained as 0.0440, 0.0550 and 0.0675 respectively. Prepare a chart as in Figure 1, with the above decision lines and plot the points c_i $(i = 1, 2, 3)$. From the Figure 1, we observe that all the points within the decision lines, hence H_0 is accepted and we may conclude that the coefficients of variation of the three different dates are the same.

5. Conclusion

Note that H_o is accepted by both Chi-Square test and the bootstrap graphical method. When H_o is rejected, chi-square test reveals the statistically significant differences among the coefficients of variation being compared, while the graphical method not only reveals the statistically significant differences but also identify the source of heterogeneity of coefficients of variation.

Figure

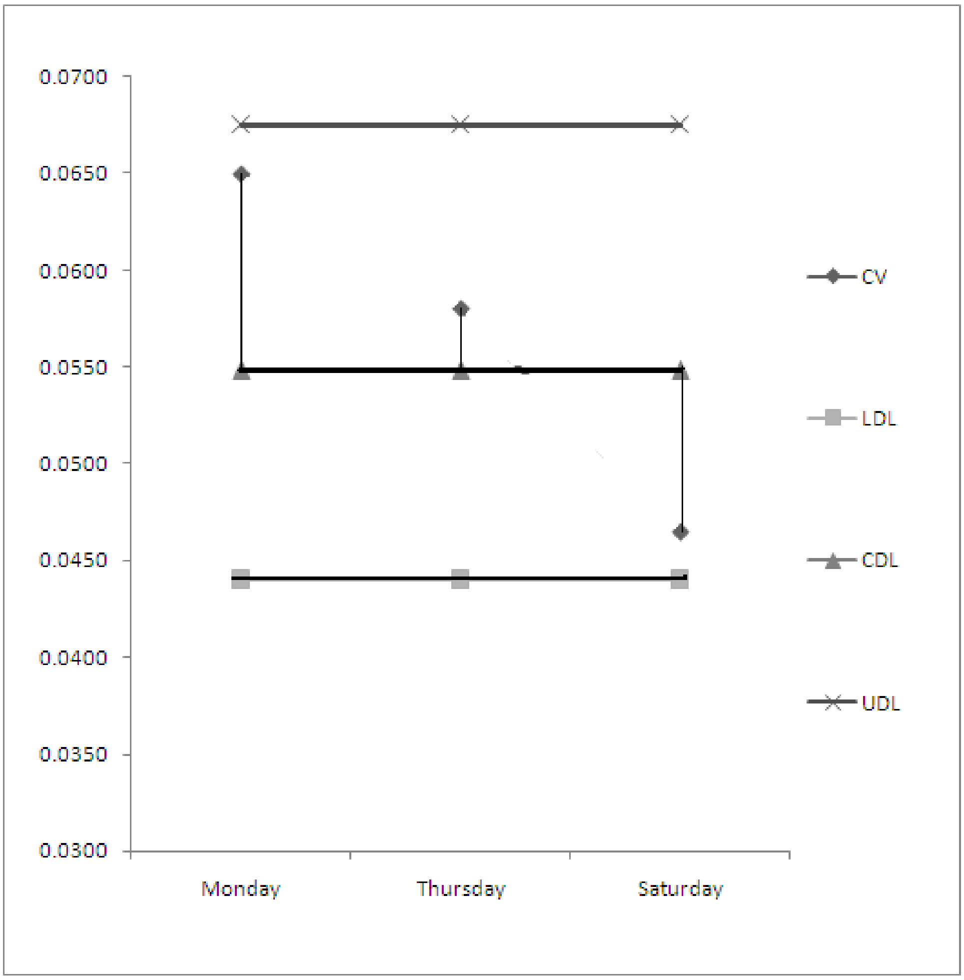

Figure 1: *Decision Lines for the Coefficients of Variation*

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Finitesimale Deformation in a Rotating Disc Having Variable Density Parameter

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Abstract Finitesimale deformation in a rotating disc having variable density parameter has been studied by using Seth's transition theory. With the effect of density variation parameter, rotating disc requires lesser angular speed for compressible as well as incompressible materials. Circumferential stresses are maximum at the internal surface for incompressible materials as compared to compressible material. Rotating disc is likely to fracture by cleavage close to the bore.
Keywords *Disc, Stresses, Deformation, Yielding, Angular Speed, Density*

1. Introduction

Disc plays an important role in machine design. Stress analysis of rotating discs has an important role in engineering design. Rotating discs are the most critical part of rotors, turbines motor, compressors, high speed gears, flywheel, sink fits, turbo jet engines and computer's disc drive etc. Solutions for thin isotropic discs can be found in most of the standard elasticity and plasticity textbooks [1, 2, 3, 4, 5]. Chakrabarty [4] and Heyman [6] solved the problem for the plastic state by utilizing the solution in the elastic state and consider the plastic range with the help of Tresca's yield condition. Further, to obtain the elastic-plastic stresses, these authors matched the elastic and plastic stresses at the same radius $r = c$ of the disc. Perfectly elasticity and ideal plasticity are two extreme properties of the material and the use of ad-hoc rule like yield condition amounts to divide the two extreme properties by a sharp line, which is not physically possible. Seth's transition theory [7] does not required any assumptions like an yield criterion, incompressibility condition, associated flow rule and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. This theory [7] utilizes the concept of generalized strain measure and asymptotic solution at critical points or turning points of the differential equations defining the deformed field and has been successfully applied to a large number of problems [7-29]. Seth [8] has defined the generalized principal strain measures as:

$$e_{ii} = \int_0^{\overset{A}{e}_{ii}} \left[1-2\overset{A}{e}_{ii}\right]^{\frac{n}{2}-1} d\,\overset{A}{e}_{ii} = \frac{1}{n}\left[1-\left(1-2\overset{A}{e}_{ii}\right)^{\frac{n}{2}}\right], (i=1, 2, 3) \quad (1)$$

Where n is the measure and $\overset{A}{e}_{ii}$ is the Almansi finite strain components. For n = -2, -1, 0, 1, 2 it gives Cauchy, Green Hencky, Swainger and Almansi measures respectively. In this research paper, we investigate the problem of finitesimale deformation in a rotating disc having variable density parameter by using Seth's transition theory. The density of disc is assumed to vary along the radius in the form:

$$\rho = \rho_0 \left(r/b\right)^{-m} \quad (2)$$

Where ρ_0 is the constant density at r = b and m *is* the density variation parameter. Results have been discussed and presented graphically.

2. Mathematical Model

We consider a thin annular disc of variable density with central bore of inner radius a and outer radius b is considered (Figure 1).

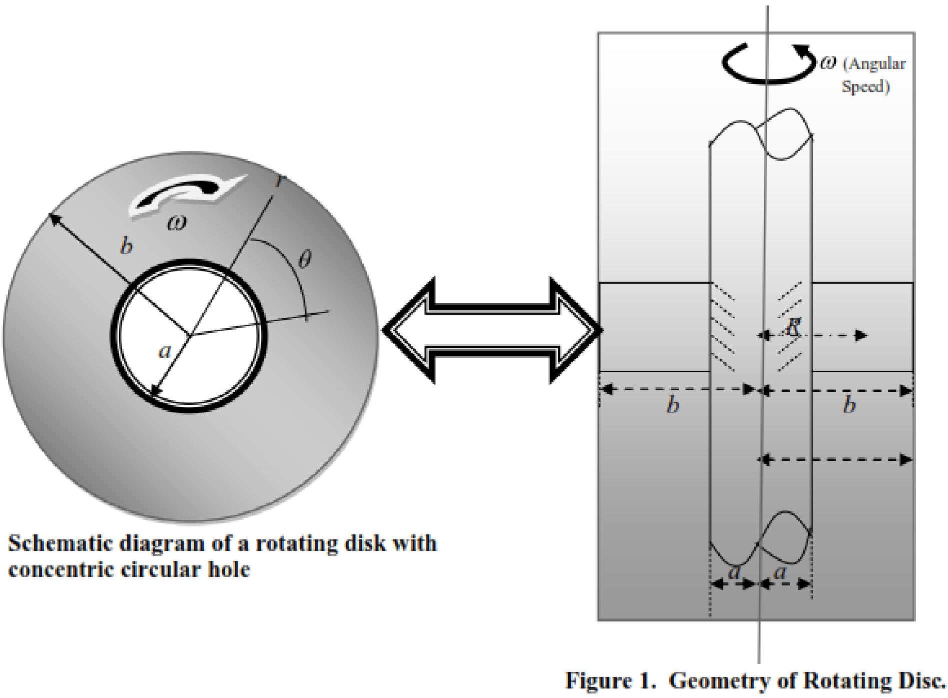

Figure 1: *Geometry of Rotating Disc.*

The disc is rotating with angular speed ω of gradually increasing magnitude about an axis perpendicular to its plane and passing through the center. The thickness of disc is assumed small so that the disc is effectively in a state of plane stress, that is, the axial stress T_{zz} is zero.

2.1 Formulation of the Problem

Displacement components in cylindrical polar co-ordinate (r,θ,z) are given by [8] as:

$u = r(1-\beta)$, v = 0, w = dz (3)

Where β is position function, depending on $r = \sqrt{x^2+y^2}$ only and d is a constant.

The finite strain components are given by [8] as:

$$\overset{A}{e}_{rr} = \frac{1}{2}\left[1-\left(r\beta'+\beta\right)^2\right];\ \overset{A}{e}_{\theta\theta} = \frac{1}{2}\left[1-\beta^2\right];\ \overset{A}{e}_{zz} = \frac{1}{2}\left[1-(1-d)^2\right];\ \overset{A}{e}_{r\theta} = \overset{A}{e}_{\theta z} = \overset{A}{e}_{zr} = 0 \quad (4)$$

Where $\beta' = d\beta / dr$ and meaning of superscripts "*A*" is Almansi.

By substituting eq. (4) in eq. (1), the generalized components of strain become:

$$e_{rr} = \frac{1}{n}\left[1-\left(r\beta'+\beta\right)^n\right]; e_{\theta\theta} = \frac{1}{n}\left[1-\beta^n\right]; e_{zz} = \frac{1}{n}\left[1-(1-d)^n\right]; e_{r\theta} = e_{\theta z} = e_{zr} = 0 \quad (5)$$

The stress–strain relations for isotropic material are given [5]:

$$T_{ij} = \lambda\delta_{ij}I_1 + 2\mu e_{ij},\ (i, j = 1, 2, 3) \quad (6)$$

Where T_{ij} are stress components, λ and μ are Lame's constants, $I_1 = e_{kk}$ is the first strain invariant, δ_{ij} is the Kronecker's delta.

Equation (6) for this problem becomes

$$T_{rr} = \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr}+e_{\theta\theta}\right]+2\mu e_{rr};\ T_{\theta\theta} = \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr}+e_{\theta\theta}\right]+2\mu e_{\theta\theta};\ T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0 \quad (7)$$

By substituting eq. (5) in eq. (7), the stresses are obtained as:

$$T_{rr} = \frac{2\mu}{n}\left[3-2C-\beta^n\left\{1-C+(2-C)(P+1)^n\right\}\right]$$

$$T_{\theta\theta} = \frac{2\mu}{n}\left[3-2C-\beta^n\left\{2-C+(1-C)(P+1)^n\right\}\right]$$

and $T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0$ (8)

where C is the compressibility factor of the material in term of Lame's constant, given by $C = 2\mu / \lambda + 2\mu$.

The equations of motion are all satisfied except:

$$\frac{d}{dr}\left(rT_{rr}\right) - T_{\theta\theta} + \rho\omega^2 r^2 = 0 \quad (9)$$

Where ρ is the density of the material of the rotating disc.

By using eqs. (8) in eq. (9), we get a non- linear differential equation for β as:

$$(2-C)n\beta^{n+1}P(P+1)^{n-1}\frac{dP}{d\beta} = \left[\frac{n\rho\omega^2 r^2}{2\mu} + \beta^n\left\{1-(P+1)^n - nP\left[1-C+(2-C)(P+1)^n\right]\right\}\right] \quad (10)$$

Where C is the compressibility factor of the material in term of Lame's constant, given by $C = 2\mu / \lambda + 2\mu$ and P is dependence function of β and β is dependence function of r only.
From eq. (10), the turning points of β are $P = -1$ and $\pm\infty$.

A. *Boundary Conditions* The boundary condition of the rotating disc is:

(i) $T_{rr} = 0$, $r = a$
(ii) $T_{rr} = 0$ $r = b$ (11)

Where T_{rr} denote stress along the radial direction.

B. *Solution of Problem* It has been shown that the asymptotic solution through the principal stress leads from elastic state to the plastic state (see Seth [7, 8], Gupta and Thakur [9-11] and Thakur Pankaj [12 - 29] at the transition point $P \to \pm\infty$. The transition function R is defined as:

$$R = \frac{nT_{\theta\theta}}{2\mu} = 3 - 2C - \beta^n \left[2 - C + (1 - C)(P + 1)^n \right] \tag{12}$$

Taking the logarithmic differentiating of eq. (12) with respect to r, we get:

$$\frac{d}{dr}(\log R) = \left(-\frac{n\beta^n P}{r} \right) \frac{\left[2 - C + (1 - C)(P + 1)^{n-1} \left\{ (P + 1) + \beta \frac{dP}{d\beta} \right\} \right]}{\left\{ 3 - 2C - \beta^n \left[2 - C + (1 - C)(P + 1)^n \right] \right\}} \tag{13}$$

By substituting the value of $dP / d\beta$ from eq. (10) into eq. (13) and by taking asymptotic value $P \to \pm\infty$, one gets after integration:

$$R = Ar^{\nu - 1} \tag{14}$$

Where A is a constant of integration, which can be determined by boundary condition and by, $\nu = 1 - C/2 - C$ is the Poisson's ratio.
From eq. (12) and (14), it follows:

$$T_{\theta\theta} = \left(\frac{2\mu}{n} \right) Ar^{\nu - 1} \tag{15}$$

By substituting eq. (15) into eq. (9) and using eq. (2), then integrating, we get:

$$T_{rr} = \frac{B}{r} + \left\{ \frac{2\mu}{n\nu} \right\} Ar^{\nu - 1} - \frac{\rho_0 \omega^2 b^m r^{2-m}}{(3 - m)} \tag{16}$$

Where B is a constant of integration, which can be determined by boundary condition.

By applying boundary condition from eq. (11) in eq. (16), we get: $A = \dfrac{\rho_0 \omega^2 n\nu b^m \left(b^{3-m} - a^{3-m} \right)}{2\mu(3 - m)\left(b^\nu - a^\nu \right)}$

and $B = \dfrac{\rho_0 \omega^2 b^m a^{3-m}}{(3 - m)} - \dfrac{\rho_0 \omega^2 b^m \left(b^{3-m} - a^{3-m} \right)}{(3 - m)\left(b^\nu - a^\nu \right)} a^\nu$.

By substituting the value of A and B into eqs. (15) and (16), we get:

$$T_{rr} = \frac{\rho_0 \omega^2 b^m}{(3-m) r} \left[\left(\frac{b^{3-m} - a^{3-m}}{b^\nu - a^\nu} \right) \left(r^\nu - a^\nu \right) - r^{3-m} + a^{3-m} \right] \quad (17)$$

$$T_{\theta\theta} = \frac{\rho_0 \omega^2 b^m \nu \left(b^{3-m} - a^{3-m} \right)}{(3-m) \left(b^\nu - a^\nu \right)} r^{\nu-1} \quad (18)$$

equations (17) and (18) gives elastic-plastic transitional stresses in a thin rotating disc of variable thickness with edge loading.

C. ***Initial Yielding of Rotating Disc*** It is seen from equ. (18) that $|T_{\theta\theta}|$ is maximum at the internal surface (*r* = *a*). Therefore yielding will take place at the inner surface and equ. (18) become:

$$|T_{\theta\theta}|_{r=a} = \left| \frac{\rho_0 \omega^2 b^m \nu \left(b^{3-m} - a^{3-m} \right)}{(3-m) \left(b^\nu - a^\nu \right)} a^{\nu-1} \right| \equiv Y(say)$$

and angular speed ω_i necessary for initial yielding is given by:

$$\Omega_i^2 = \frac{\rho_0 \omega_i^2 b^2}{Y} = \frac{(3-m) \left(b^\nu - a^\nu \right) b^2}{\nu \left(b^{3-m} - a^{3-m} \right) b^m a^{\nu-1}} \quad (19)$$

Where $\omega_i = \frac{1}{b} \Omega_i \left(Y / \rho_0 \right)^{\frac{1}{2}}$. We introduce the following non-dimensional components as: *R* = *r*/*b*, $R_0 = a/b, \Omega^2 = \rho_0 \omega^2 b^2 / Y, \sigma_r = T_{rr} / Y, \sigma_\theta = T_{\theta\theta} / Y$. Eqs. (17), (18) and (19) become:

$$\sigma_r = \frac{\Omega_i^2}{(3-m) R} \left[\frac{\left(1 - R_0^{3-m} \right)}{\left(1 - R_0^\nu \right)} \left(R^\nu - R_0^\nu \right) - R^{3-m} + R_0^{3-m} \right] \quad (20)$$

$$\sigma_\theta = \frac{\Omega_i^2 \nu \left(1 - R_0^{3-m} \right)}{(3-m) \left(1 - R_0^\nu \right)} R^{\nu-1} \quad (21)$$

$$\Omega_i^2 = \frac{(3-m) \left(1 - R_0^\nu \right)}{\left(1 - R_0^{3-m} \right) \nu} R_0^{1-\nu} \quad (22)$$

D. ***Fully Plastic State of Rotating Disc*** The angular speed $\omega_f > \omega_i$ for which the rotating disc become fully plastic $(\nu \to 1/2 = 0.5)$ at the external surface *r* = b, equation (18) becomes

$$|T_{\theta\theta}|_{r=b} = \left| \frac{\rho_0 \omega^2 b^m \left(b^{3-m} - a^{3-m} \right)}{2(3-m) \sqrt{b} \left(\sqrt{b} - \sqrt{a} \right)} \right| \equiv Y^*(say)$$

and angular speed ω_f necessary for initial yielding is given by:

$$\Omega_f^2 = \frac{\rho_0 \omega_f^2 b^2}{Y^*} = \frac{2(3-m)\sqrt{b}\left(\sqrt{b}-\sqrt{a}\right)b^2}{b^m\left(b^{3-m}-a^{3-m}\right)} \quad (23)$$

and stresses and angular speed give by eqs. (20), (21) and (23) for fully plastic state $(\nu \to 1/2 = 0.5)$ become:

$$\sigma_r = \frac{\Omega_f^2}{(3-m)R}\left[\frac{\left(1-R_0^{3-m}\right)}{\left(1-\sqrt{R_0}\right)}\left(\sqrt{R}-\sqrt{R_0}\right)-R^{3-m}+R_0^{3-m}\right] \quad (24)$$

$$\sigma_\theta = \frac{\Omega_f^2\left(1-R_0^{3-m}\right)}{2(3-m)\sqrt{R}\left(1-\sqrt{R_0}\right)} \quad (25)$$

$$\Omega_f^2 = \frac{\rho_0 \omega_f^2 b^2}{Y^*} = \frac{2(3-m)\left(1-\sqrt{R_0}\right)}{\left(1-R_0^{3-m}\right)} \quad (26)$$

Neglects density parameter (*m* = 0), transitional stresses and angular speed from eqs. (20)- (22) becomes:

$$\sigma_r = \frac{\Omega_i^2}{3R}\left[\frac{\left(1-R_0^3\right)}{\left(1-R_0^\nu\right)}\left(R^\nu-R_0^\nu\right)-R^3+R_0^3\right] \quad (27)$$

$$\sigma_\theta = \frac{\Omega_i^2 \nu\left(1-R_0^3\right)}{3\left(1-R_0^\nu\right)}R^{\nu-1} \quad (28)$$

$$\Omega_i^2 = \frac{3\left(1-R_0^\nu\right)}{\left(1-R_0^3\right)\nu}R_0^{1-\nu} \quad (29)$$

and without density variation parameter, stresses and angular speed for fully plastic state from eqs. (24), (25) and (26) becomes:

$$\sigma_r = \frac{\Omega_f^2}{3R}\left[\frac{\left(1-R_0^3\right)}{\left(1-\sqrt{R_0}\right)}\left(\sqrt{R}-\sqrt{R_0}\right)-R^3+R_0^3\right] \quad (30)$$

$$\sigma_\theta = \frac{\Omega_f^2\left(1-R_0^3\right)}{6\sqrt{R}\left(1-\sqrt{R_0}\right)} \quad (31)$$

$$\Omega_f^2 = \frac{\rho_0 \omega_f^2 b^2}{Y^*} = \frac{6\left(1-\sqrt{R_0}\right)}{\left(1-R_0^{\,3}\right)} \quad (32)$$

3. Numerically Discussion

For calculating the stresses based on the above analysis, the following values have been taken as *C* = 0.00, 0.25, 0.5, 0.75, *m* = 0, 1, 2 respectively. Curves have been drawn in figure 2 between angular speed Ω_i^2 required for initial yielding and various radii ratios $R_0 = a/b$ for C = 0, 0.25, 0.5 at m = 0, 1, 2. It has been observed that the rotating disc made of incompressible material required higher angular speed for initial yielding as compared to disc made of compressible materials. With effect of density variation parameter, rotating disc requires lesser angular speed as compared to without density variation parameter. It can also be seen from Table 1 that for compressible material higher percentage increased in angular speed is required to become fully plastic as compared to rotating disc made of incompressible material.

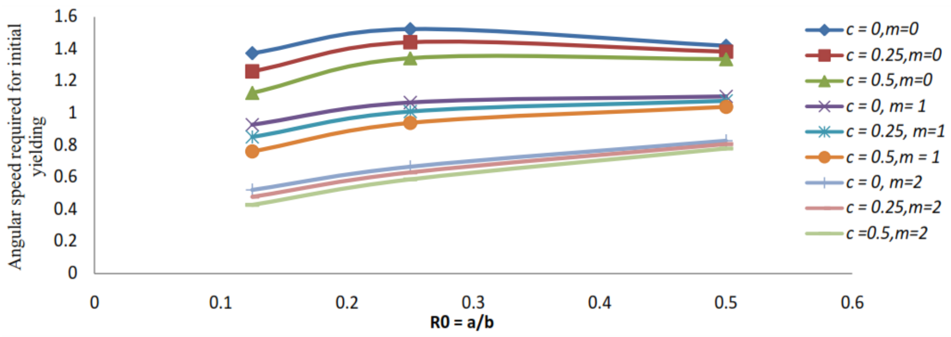

Figure 2: *Angular speed required for Initial Yielding State along the Radii Ratio Ro = a/b.*

Table 1: *Angular Speed Required for Initial Yielding and Fully Plastic State*

	Variable density parameter *m*	**Compressibility of Material C**	**Angular Speed required for initial yielding (Ω_i^2)**	**Angular Speed required for fully-plastic state(Ω_f^2)**	**Percentage increase in Angular speed** $\left(\sqrt{\frac{\Omega_f^2}{\Omega_i^2}}-1\right)\times 100$
$0.5 \le R \le 1.0$	0	0	1.420161	2.008411	18.92071307%
	1	0	1.104569	1.562097	18.92073204%
	2	0	0.828427	1.171573	18.92072679%
	0	0.25	1.383601	2.008411	20.48163765%
	1	0.25	1.076134	1.562097	20.48162673%
	2	0.25	0.8071	1.171573	20.48167691%
	0	0.5	1.336737	2.008411	22.57539772%
	1	0.5	1.039684	1.562097	22.57539993%
	2	0.5	0.779763	1.171573	22.57541301%

In figures 3 and 4, curve have been drawn between stresses and radii ratio *R* = *r/b* for elastic-plastic transition state and fully plastic state. It has been seen circumferential stresses is maximum at the internal surface for incompressible materials (*C* = 0) as compared to compressible materials (*C* = 0.25, 0.5). Density variation parameter has a quit effect on circumferential stresses *i.e*. with the introduction of density variation parameter it decreases the values of circumferential stresses at the internal at the internal surface for transitional state and for fully plastic state. Rotating disc *is* likely to fracture by cleavage close to the bore.

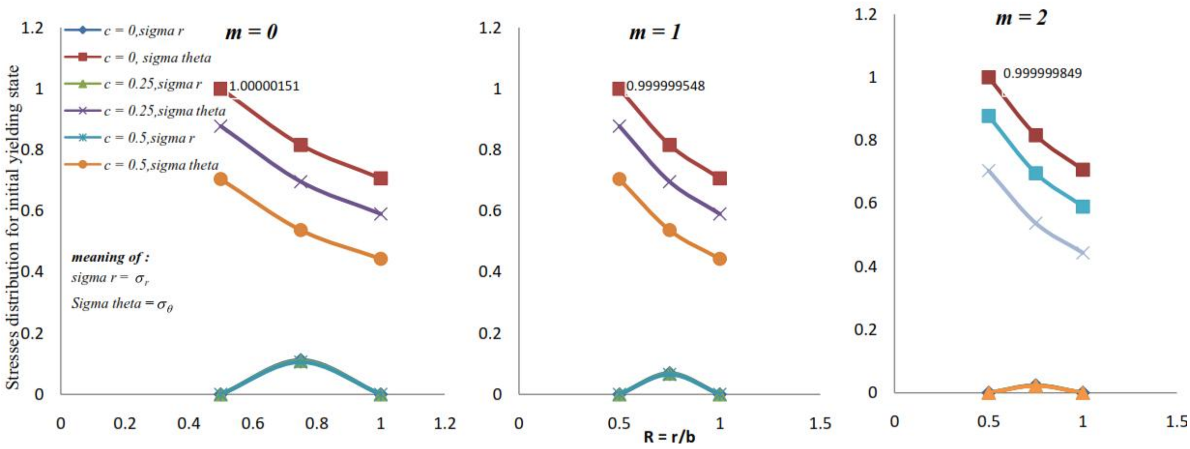

Figure 3: *Stresses Distribution in a Thin Rotating Disc for Initial Yielding State along the Radius Ratio R = r/b.*

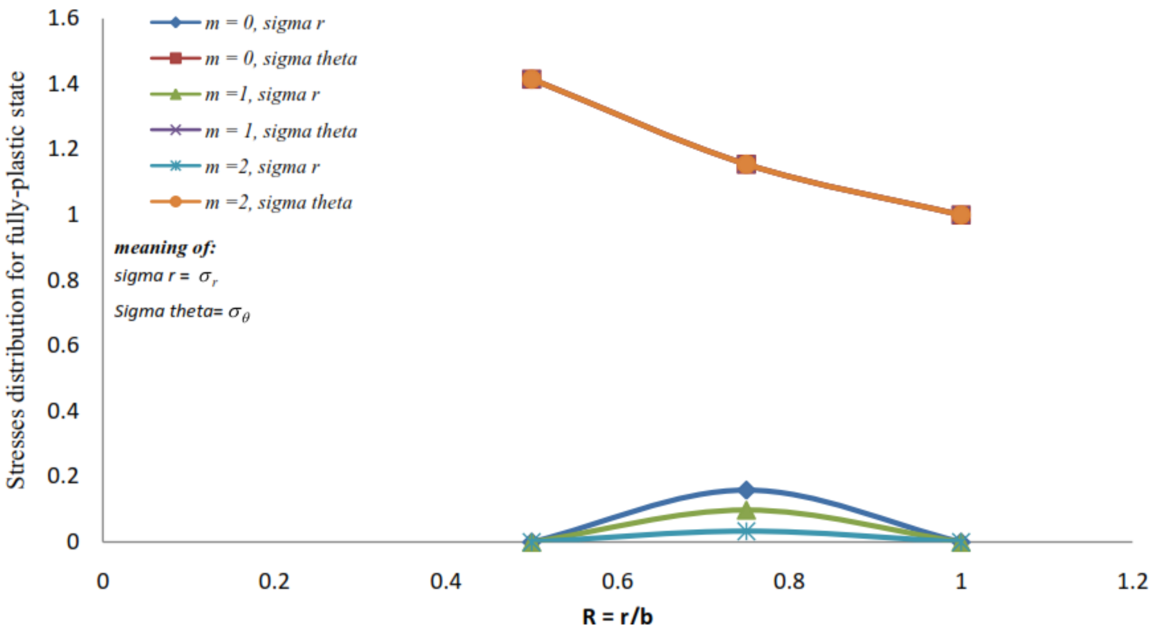

Figure 4: *Stresses Distribution in a Thin Rotating Disc for Fully-Plastic State along the Radius Ratio R = r/b.*

4. Conclusion

It has been observed that with the effect of density variation parameter, rotating disc requires lesser angular speed for compressible as well as incompressible material. Circumferential stresses are maximum at the internal surface for incompressible materials as compared to compressible material. Rotating disc is likely to fracture by cleavage close to the bore.

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Research Article **Open Access**

The Product of Bessel Functions of the First Kind with Fractional Calculus Operators Involving Appell Hypergeometric Function

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Abstract Two integral transforms involving the function F_3 occurring in the Kernels are considered. The function F_3 is the familiar Appell Hypergeometric function. They generalize the Saigo and classical Riemann-Liouville fractional integral operators. Formulas for compositions of such generalized fractional integrals with the product of Bessel functions of the first kind are proved. Special cases for the product of cosine and sine functions are given. The results are established in terms of generalized Lauricella function due to Srivastava and Daoust. Corresponding assertions for the Saigo and Riemann-Liouville fractional integrals are presented. (AMS subject classification 33C10, 33C45, 26C33).

Keywords *Fractional Integrals, Bessel Functions of the First Kind, Generalized Hypergeometric Series, Generalized Lauricella Series in Several Variables, Appell Function-* F_3*, Cosine and Sine Trigonometric Functions*

1. Introduction

This paper deals with two generalized fractional calculus operators involving Appell function- F_3 are defined by Saigo and Maeda [15] where $\alpha,\alpha',\beta,\beta',\gamma \in C$ and $x>0$ $\left(\mathrm{Re}(\gamma)>0\right)$ by

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-t/x,1-x/t\right) f(t)\,dt \quad \mathrm{Re}(\gamma)>0 \tag{1.1}$$

where the function $f(x)$ is analytic in a simply-connected region of the complex x-plane containing the origin and it is understood that $(x-t)$ denotes the principal value for $0\le \arg(x-t)<2\pi$.

and
$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-x/t,1-t/x\right) f(t)\,dt \quad \mathrm{Re}(\gamma)>0 \tag{1.2}$$

Here, $\Gamma(\gamma)$ is the Euler gamma function [1, Section 1],

$\mathrm{Re}(\gamma)$ denotes the real part of γ, and $F_3(\alpha,\alpha',\beta,\beta';\gamma;z,\zeta)$ is the familiar Appell hypergeometric function defined by:

$$F_3(\alpha,\alpha',\beta,\beta';\gamma;z,\xi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{z^m}{m!}\frac{\xi^n}{n!} \qquad (|z|<1,|\xi|<1) \tag{1.3}$$

where $(z)_m$ and $(z)_n$ are the Pochhammer symbol defined by $z\in C$ and $m,n\in N_0=N\cup\{0\}$, $N=\{1,2,3,...\}$ by

$$(z)_0=1,\ (z)_m=z(z+1)...(z+m-1) \text{ or } (z)_n=z(z+1)...(z+n-1) \quad (m,n\in N) \tag{1.4}$$

The series in (1.3) is absolutely convergent for

$$(|z|<1,\ |\xi|<1) \text{ and } (|z|=1,\ |\xi|=1) \qquad \text{where } (z,\xi\neq 1) \tag{1.5}$$

Operators (1.1) and (1.2) were defined by Saigo and Maeda [15] and their properties were investigated by many authors, see bibliography.

1. Definition

It may be observed that from (1.1) for $\alpha=\lambda+\mu,\quad \alpha'=\beta'=0,\quad \beta=-\eta,\quad \gamma=\lambda$ (1.6)

we obtain the relationship

$$I_{0,x}^{\lambda+\mu,0,-\eta,0,\lambda}=I_{0,x}^{\lambda,\mu,\eta} \tag{1.7}$$

where $$I_{0,x}^{\lambda,\mu,\eta}\ f(x)=\frac{x^{-\lambda-\mu}}{\Gamma(\lambda)}\int_0^x (x-t)^{\lambda-1}\ {}_2F_1\left(\lambda+\mu,-\eta;\lambda;1-\frac{t}{x}\right)f(t)\ dt \tag{1.8}$$

in terms of the Saigo type fractional integral operator $I_{0,x}^{\lambda,\mu,\eta}$ [9], and this reduces to the Riemann-Liouville fractional integrals if we put $\mu=-\lambda$ [16], then

$$I_{0,x}^{\lambda,-\lambda,\eta}\ f(x)=I_{0,x}^{\lambda}\ f(x)=\frac{1}{\Gamma(\lambda)}\int_0^x (x-t)^{\lambda-1}\ f(t)\ dt \qquad (x>0) \tag{1.9}$$

We investigate compositions of integral transforms (1.1) and (1.2) with the product of Bessel function of the first kind $J_v(z)$, which is defined for complex $z\in C\ (z\neq 0)$ and $v\in C\ (\mathrm{Re}(v)>-1)$ by [2, 7.2 (2)].

$$J_v(z)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{v+2k}}{\Gamma(v+k+1)\ k!} \tag{1.10}$$

We prove that such compositions are expressed in terms of the generalized Lauricella function due to Srivastava and Daoust. [6], which is defined by

$$F_{C:D';...;D^{(n)}}^{A:B';...;B^{(n)}}\begin{bmatrix} z_1\\ \vdots\\ z_n\end{bmatrix}=F_{C:D';...;D^{(n)}}^{A:B';...;B^{(n)}}\left[\begin{matrix}\left[(a):\theta',...,\theta^{(n)}\right],\left[(b'):\phi'\right];...;\left[(b)^{(n)}:\phi^{(n)}\right];\\ \left[(c):\psi',...,\psi^{(n)}\right],\left[(d'):\delta'\right];...;\left[(d)^{(n)}:\delta^{(n)}\right];\end{matrix}\ z_1,...,z_n\right]$$

$$= \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{k_1\theta'_j+\ldots+k_n\theta_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{k_1\psi'_j+\ldots+k_n\psi_j^{(n)}}} \frac{\prod_{j=1}^{B'}(b'_j)_{k_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{k_n\phi_j^{(n)}}}{\prod_{j=1}^{D'}(d'_j)_{k_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{k_n\delta_j^{(n)}}} \frac{z_1^{k_1}}{k_1!}\cdots\frac{z_n^{k_n}}{k_n!} \tag{1.11}$$

the coefficients

$$\left\{ \begin{array}{l} \theta_j^{(k)}\ (j=1,\ldots,A);\quad \phi_j^{(k)}\ \left(j=1,\ldots,B^{(k)}\right) \\ \psi_j^{(k)}\ (j=1,\ldots,C);\ \delta_j^{(k)}\ \left(j=1,\ldots,D^{(k)}\right);\quad \forall k\in\{1,\ldots,n\} \end{array} \right\} \tag{1.12}$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\ldots,a_A$, $\left(b^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}\ \left(j=1,\ldots,B^{(k)}\right)\ \ \forall k\in\{1,\ldots,n\}$, with similar interpretations for (c) and $\left(d^k\right)\ \ (k=1,2,\ldots,n)$.

$(z)_n$ is a generalized of the Pochhammer symbol (1.4)

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(n)},\qquad (z,n\in C) \tag{1.13}$$

The multiple series (1.11) converges absolutely either

(i) $\Delta_j > 0\ \ (i=1,\ldots,n)\qquad \forall z_1,\ldots,z_n\in C,$ or

(ii) $\Delta_j = 0\ \ (i=1,\ldots,n),\ \ \forall z_1,\ldots,z_n\in C,\ \ |z_i|<\varsigma_i\quad (i=1,\ldots,n)$ and divergent when

$\Delta_i < 0\ \ (i=1,\ldots,n)$; except for the trivial case $z_1=\ldots..=z_n=0$, where

$$\Delta_i = 1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}\qquad (i=1,\ldots,n) \tag{1.14}$$

$$\varsigma_i = \min_{\mu_i,\ldots,\mu_n>0}\{E_i\}\qquad (i=1,\ldots,n) \tag{1.15}$$

with

$$1+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}\ \frac{\left\{\prod_{j=1}^{C}\left(\sum_{j=1}^{n}\mu_i\psi_j^{(i)}\right)^{\psi_j^{(i)}}\right\}\left\{\prod_{j=1}^{D^{(i)}}\left(\delta_j^{(i)}\right)^{\delta_j^{(i)}}\right\}}{\left\{\prod_{j=1}^{A}\left(\sum_{j=1}^{n}\mu_i\theta_j^{(i)}\right)^{\theta_j^{(i)}}\right\}\left\{\prod_{j=1}^{B^{(i)}}\left(\phi_j^{(i)}\right)^{\phi_j^{(i)}}\right\}} \tag{1.16}$$

For more details see [6].Special cases of (1.11) are established in terms of generalized hypergeometric function of one and two variables respectively, for the sake of completeness we define these functions here. A generalized hypergeometric function ${}_pF_q(z)$ is defined for complex $a_i,b_j\in C,\ \ b_j\neq 0,-1,\ldots.\quad (i=1,2,\ldots,p;\ j=1,2,\ldots,q)$ by the generalized hypergeometric series [1, 4.1(1)]:

$${}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!} \tag{1.17}$$

The series is absolutely convergent for all values of $z \in C$ if $p \le q;$ and it is an entire function of z. We define a generalization of the Kampé de Fériet function by means of the double hypergeometric series [6].

$$F_{l:m;n}^{p:q;k}\left[\begin{matrix}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{matrix}\; x, y\right] =$$

$$= \sum_{r,s=0}^{\infty} \frac{\left\{\prod_{j=1}^{p}(a_j)_{r+s}\right\}\left\{\prod_{j=1}^{q}(b_j)_r\right\}\left\{\prod_{j=1}^{k}(c_j)_s\right\}}{\left\{\prod_{j=1}^{l}(\alpha_j)_{r+s}\right\}\left\{\prod_{j=1}^{m}(\beta_j)_r\right\}\left\{\prod_{j=1}^{n}(\gamma_j)_s\right\}} \frac{x^r}{r!}\frac{y^s}{s!} \tag{1.18}$$

This paper is organized as follows. Formulas for compositions of integral transforms (1.1) and (1.2) with the product of Bessel function (1.10) are proved in terms of generalized Lauricella function (1.11) in Section 2 and 3 respectively.

The corresponding results for the Saigo fractional integrals (1.8) and Riemann-Liouville fractional integrals (1.9) are also presented in Section 2 and 3. Special cases giving compositions of fractional integrals with the product of cosine and sine functions are considered in Section 4.

2. Left-Sided Fractional Integration of Bessel Functions

The results, given in Section 2 and 3 are based on the preliminary assertions giving composition formulas of generalized fraction integrals (1.1) and (1.2) with a power function.

Lemma 1. [See 13, equs. (2.11) and (2.12)] Let $\alpha, \alpha', \beta, \beta', \gamma \in C.$

(i) If $\mathrm{Re}(\gamma) > 0,\ \ \mathrm{Re}(\rho) > \max\left[0, \mathrm{Re}(\alpha + \alpha' + \beta' - \gamma), \mathrm{Re}(\alpha' - \beta')\right]$, then

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = x^{\rho-\alpha-\alpha'+\gamma-1} \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)\Gamma(\rho+\beta')} \tag{2.1}$$

(ii) If $\mathrm{Re}(\gamma) > 0,\ \ \mathrm{Re}(\rho) < 1 + \min\left[\mathrm{Re}(-\beta), \mathrm{Re}(\alpha + \alpha' - \gamma), \mathrm{Re}(\alpha + \beta' - \gamma)\right]$, then

$$I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = x^{\rho-\alpha-\alpha'+\gamma-1} \frac{\Gamma(1+\alpha+\alpha'-\gamma-\rho)\Gamma(1+\alpha+\beta'-\gamma-\rho)\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\Gamma(1+\alpha-\beta-\rho)} \tag{2.2}$$

The generalized left-sided fractional integration (1.1) of the product of Bessel function (1.10) is given by the following result.

Theorem 1.

Let $n \in N,\ \alpha, \alpha', \beta, \beta', \gamma, \sigma, v_j \in C$ and $a_j, \rho_j \in R_+\ \ (j = 1, 2, \ldots, n)$ be such that

$$\mathrm{Re}(\gamma) > 0,\ \mathrm{Re}(v_j) > -1,\ \ \mathrm{Re}\left(\sigma + \sum_{j=1}^{n} \rho_j v_j\right) > \max\left[0, \mathrm{Re}(\alpha+\alpha'+\beta'-\gamma), \mathrm{Re}(\alpha'-\beta')\right] \tag{2.3}$$

Then there holds the formula

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}\prod_{j=1}^{n}J_{v_j}(a_j t^{\rho_j})\right]\right)(x) = x^{\sigma-\alpha-\alpha'+\gamma-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{a_j x^{\rho_j}}{2}\right)^{v_j}}{\Gamma(v_j+1)}\right)\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(o)\Gamma(p)\Gamma(q)}$$

$$\times F_{3:1,\ldots,1}^{3:0,\ldots,0}\left[\begin{matrix}[l:2\rho_1,\ldots,2\rho_n],[m:2\rho_1,\ldots,2\rho_n],[n:2\rho_1,\ldots,2\rho_n]: \\ [o:2\rho_1,\ldots,2\rho_n],[p:2\rho_1,\ldots,2\rho_n],[q:2\rho_1,\ldots,2\rho_n]:[v_1+1:1],\ldots,[v_n+1:1]:\end{matrix};-\frac{a_1^2x^{2\rho_1}}{4},\ldots,-\frac{a_n^2x^{2\rho_n}}{4}\right]; \qquad (2.4)$$

where $l=\sigma+\sum_{j=1}^{n}(v_j\rho_j)$, $m=\sigma+\gamma-\alpha-\alpha'-\beta+\sum_{j=1}^{n}(v_j\rho_j)$,

$n=\sigma+\beta'-\alpha'+\sum_{j=1}^{n}(v_j\rho_j)$, $o=\sigma+\gamma-\alpha-\alpha'+\sum_{j=1}^{n}(v_j\rho_j)$,

$p=\sigma+\gamma-\alpha'-\beta+\sum_{j=1}^{n}(v_j\rho_j)$, $q=\sigma+\beta'+\sum_{j=1}^{n}(v_j\rho_j)$, and $F_{3:1,\ldots,1}^{3:0,\ldots,0}[.]$ is given by (1.11).

Proof: First of all we note that Δ_i in (1.14) is given by $\Delta_i=1+n>0\ \ (i=1,\ldots,n\in N)$, and therefore $F_{3:1,\ldots,1}^{3:0,\ldots,0}[.]$ in the right hand side of (2.4) is defined. Now we prove (2.4). Applying equation (1.10) by using (1.1) and (1.11) with changing the order of integration and summation, we get

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}\prod_{j=1}^{n}J_{v_j}(a_j t^{\rho_j})\right]\right)(x)$$

$$=\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}\left(\sum_{k_1=0}^{\infty}\frac{(-1)^{k_1}\left(\frac{a_1t^{\rho_1}}{2}\right)^{v_1+2k_1}}{\Gamma(v_1+k_1+1)\,k_1!}\right)\cdots\cdots\left(\sum_{k_n=0}^{\infty}\frac{(-1)^{k_n}\left(\frac{a_nt^{\rho_n}}{2}\right)^{v_n+2k_n}}{\Gamma(v_n+k_n+1)\,k_n!}\right)\right]\right)(x)$$

$$=\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{(-1)^{k_1}\left(\frac{a_1}{2}\right)^{v_1+2k_1}}{\Gamma(v_1+1)\,(v_1+1)_{k_1}\,k_1!}\cdots\cdots\frac{(-1)^{k_n}\left(\frac{a_n}{2}\right)^{v_n+2k_n}}{\Gamma(v_n+1)\,(v_n+1)_{k_n}\,k_n!}$$

$$\times\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left\{t^{\sigma+(v_1\rho_1+\ldots+v_n\rho_n)+(2\rho_1k_1+\ldots+2\rho_nk_n)-1}\right\}\right)(x).$$

By using (2.3), for any $k_j\in N_0\ \ (j=1,\ldots,n)$

$$\mathrm{Re}\left(\sigma+\sum_{j=1}^{n}v_j\rho_j+2\sum_{j=1}^{n}\rho_jk_j\right)\geq\mathrm{Re}\left(\sigma+\sum_{j=1}^{n}v_j\rho_j\right)>\max\left[0,\mathrm{Re}(\alpha+\alpha'+\beta'-\gamma),\mathrm{Re}(\alpha'-\beta')\right].$$

Applying Lemma 1(i) and using (2.1) with x and ρ are replaced by t and

$\left(\sigma+\sum_{j=1}^{n}\rho_jv_j+2\sum_{j=1}^{n}\rho_jk_j\right)$ respectively for $j=1,2,\ldots,n$. Then we obtain

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}\prod_{j=1}^{n}J_{v_j}(a_j t^{\rho_j})\right]\right)(x)$$

$$=\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{(-1)^{k_1}\left(\frac{a_1}{2}\right)^{v_1+2k_1}}{\Gamma(v_1+1)\,(v_1+1)_{k_1}\,k_1!}\cdots\cdots\frac{(-1)^{k_n}\left(\frac{a_n}{2}\right)^{v_n+2k_n}}{\Gamma(v_n+1)\,(v_n+1)_{k_n}\,k_n!}$$

$$\times \frac{\Gamma\left(\sigma+\sum_{j=1}^{n}\left(v_j\rho_j+2\rho_j k_j\right)\right)\,\Gamma\left(\sigma+\gamma-\alpha-\alpha'-\beta+\sum_{j=1}^{n}\left(v_j\rho_j+2\rho_j k_j\right)\right)}{\Gamma\left(\sigma+\gamma-\alpha-\alpha'+\sum_{j=1}^{n}\left(v_j\rho_j+2\rho_j k_j\right)\right)\,\Gamma\left(\sigma+\gamma-\alpha'-\beta+\sum_{j=1}^{n}\left(v_j\rho_j+2\rho_j k_j\right)\right)}$$

$$\times \frac{\Gamma\left(\sigma+\beta'-\alpha'+\sum_{j=1}^{n}\left(v_j\rho_j+2\rho_j k_j\right)\right)}{\Gamma\left(\sigma+\beta'+\sum_{j=1}^{n}\left(v_j\rho_j+2\rho_j k_j\right)\right)}\quad x^{\sigma-\alpha-\alpha'+\gamma-1+\sum_{j=1}^{n}\left(v_j\rho_j+2\rho_j k_j\right)}$$

$$=x^{\sigma-\alpha-\alpha'+\gamma-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{a_j x^{\rho_j}}{2}\right)^{v_j}}{\Gamma(v_j+1)}\right)\frac{\Gamma\left(\sigma+\sum_{j=1}^{n}(v_j\rho_j)\right)\,\Gamma\left(\sigma+\gamma-\alpha-\alpha'-\beta+\sum_{j=1}^{n}(v_j\rho_j)\right)\Gamma\left(\sigma+\beta'-\alpha'+\sum_{j=1}^{n}(v_j\rho_j)\right)}{\Gamma\left(\sigma+\gamma-\alpha-\alpha'+\sum_{j=1}^{n}(v_j\rho_j)\right)\,\Gamma\left(\sigma+\gamma-\alpha'-\beta+\sum_{j=1}^{n}(v_j\rho_j)\right)\,\Gamma\left(\sigma+\beta'+\sum_{j=1}^{n}(v_j\rho_j)\right)}$$

$$\times\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{\left(\sigma+\sum_{j=1}^{n}v_j\rho_j\right)_{2\rho_1k_1+\ldots+2\rho_nk_n}\left(\sigma+\gamma-\alpha-\alpha'-\beta+\sum_{j=1}^{n}v_j\rho_j\right)_{2\rho_1k_1+\ldots+2\rho_nk_n}\left(\sigma+\beta'-\alpha'+\sum_{j=1}^{n}v_j\rho_j\right)_{2\rho_1k_1+\ldots+2\rho_nk_n}}{\left(\sigma+\gamma-\alpha-\alpha'+\sum_{j=1}^{n}v_j\rho_j\right)_{2\rho_1k_1+\ldots+2\rho_nk_n}\left(\sigma+\gamma-\alpha'-\beta+\sum_{j=1}^{n}v_j\rho_j\right)_{2\rho_1k_1+\ldots+2\rho_nk_n}\left(\sigma+\beta'+\sum_{j=1}^{n}v_j\rho_j\right)_{2\rho_1k_1+\ldots+2\rho_nk_n}}$$

$$\times\frac{1}{(v_1+1)_{k_1}\ldots.(v_n+1)_{k_n}}\frac{\left(-\frac{a_1^2x^{2\rho_1}}{4}\right)^{k_1}}{k_1!}\ldots\ldots\frac{\left(-\frac{a_n^2x^{2\rho_n}}{4}\right)^{k_n}}{k_n!}\;.$$

(By using Pochhammer symbol $(a)_n=\dfrac{\Gamma(a+n)}{\Gamma(a)}$ or $\Gamma(a).(a)_n=\Gamma(a+n)$).

In compliance with equation (1.11), we get the result in (2.4). This completes the proof of the theorem. Now, if we follow Theorem 1 in respective case $\alpha'=\beta'=0,\ \beta=-\eta,\ \alpha=\alpha+\beta,\ \gamma=\alpha$. Then we have following result:

Corollary 1.1.

Let $n\in N,\ \alpha,\beta,\eta,\sigma,v_j\in C$ and $a_j,\rho_j\in R_+\ (j=1,2,\ldots..,n)$

be such that $\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(v_j)>-1,\ \mathrm{Re}\left(\sigma+\sum_{j=1}^{n}\rho_jv_j\right)>\max\left[0,\mathrm{Re}(\beta-\eta)\right]$. Then,

$$\left(I_{0,x}^{\alpha,\beta,\eta}\left[t^{\sigma-1}\prod_{j=1}^{n}J_{v_j}(a_jt^{\rho_j})\right]\right)(x)=x^{\sigma-\beta-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{a_jx^{\rho_j}}{2}\right)^{v_j}}{\Gamma(v_j+1)}\right)\frac{\Gamma(l)\Gamma(m)}{\Gamma(n)\Gamma(p)}$$

$$\times F_{2:1,\ldots,1}^{2:0,\ldots,0}\left[\begin{matrix}[l:2\rho_1,\ldots,2\rho_n],[m:2\rho_1,\ldots,2\rho_n]:\\ [n:2\rho_1,\ldots,2\rho_n],[p:2\rho_1,\ldots,2\rho_n]:[v_1+1:1],\ldots\ldots,[v_n+1:1]:\end{matrix};-\frac{a_1^2x^{2\rho_1}}{4},\ldots,-\frac{a_n^2x^{2\rho_n}}{4}\right] \quad (2.5)$$

where

$l=\sigma+\sum_{j=1}^{n}\rho_jv_j$, $m=\sigma+\eta-\beta+\sum_{j=1}^{n}\rho_jv_j$,

$n=\sigma-\beta+\sum_{j=1}^{n}\rho_jv_j$, $p=\sigma+\alpha+\eta+\sum_{j=1}^{n}\rho_jv_j$, and $F_{2:1,\ldots,1}^{2:0,\ldots,0}[.]$ is given by (1.11).

Now, if we put $\beta=-\alpha$ in Corollary 1.1 then we have following result:

Corollary 1.2.

Let $n \in N,\ \alpha, \sigma, v_j \in C$ and $a_j, \rho_j \in R_+\ (j = 1, 2, \ldots\ldots, n)$

be such that $\mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(v_j) > -1\ \ and\ \ \mathrm{Re}\left(\sigma + \sum_{j=1}^{n} \rho_j v_j\right) > 0$. Then,

$$\left({}_0I_x^{\alpha}\left[t^{\sigma-1}\prod_{j=1}^{n} J_{v_j}(a_j t^{\rho_j})\right]\right)(x) = x^{\sigma+\alpha-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{a_j x^{\rho_j}}{2}\right)^{v_j}}{\Gamma(v_j+1)}\right)\frac{\Gamma\left(\sigma+\sum_{j=1}^{n}\rho_j v_j\right)}{\Gamma\left(\sigma+\alpha+\sum_{j=1}^{n}\rho_j v_j\right)}$$

$$\times F_{1:1,\ldots,1}^{1:0,\ldots,0}\left[\begin{matrix}\left[\sigma+\sum_{j=1}^{n}\rho_j v_j : 2\rho_1,\ldots,2\rho_n\right]: \\ \left[\sigma+\alpha+\sum_{j=1}^{n}\rho_j v_j : 2\rho_1,\ldots,2\rho_n\right]:[v_1+1:1],\ldots\ldots,[v_n+1:1]:\end{matrix} ; -\frac{a_1^2 x^{2\rho_1}}{4},\ldots,-\frac{a_n^2 x^{2\rho_n}}{4}\right] \tag{2.6}$$

Corollary 1.3.

Let $\alpha, \alpha', \beta, \beta', \gamma, \sigma, v_1, v_2 \in C$ and $a_1, a_2, \rho_1, \rho_2 \in R_+$

be such that $\mathrm{Re}(\gamma) > 0,$

$\mathrm{Re}(v_1) > -1,\ \mathrm{Re}(v_2) > -1,\quad \mathrm{Re}(\sigma + \rho_1 v_1 + \rho_2 v_2) > \max\left[0, \mathrm{Re}(\alpha + \alpha' + \beta' - \gamma), \mathrm{Re}(\alpha' - \beta')\right].$

Then, we have

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}J_{v_1}(t)J_{v_2}(t)\right]\right)(x) = \frac{x^{d-1}}{2^{v_1+v_2}}\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)\Gamma(v_1+1)\Gamma(v_2+1)}$$

$$\times F_{6:1,\ldots,1}^{6:0,\ldots,0}\left[\begin{matrix}\left[\frac{a}{2}:1,1\right],\left[\frac{a+1}{2}:1,1\right],\left[\frac{b}{2}:1,1\right],\left[\frac{b+1}{2}:1,1\right],\left[\frac{c}{2}:1,1\right],\left[\frac{c+1}{2}:1,1\right]: \\ \left[\frac{d}{2}:1,1\right],\left[\frac{d+1}{2}:1,1\right],\left[\frac{e}{2}:1,1\right],\left[\frac{e+1}{2}:1,1\right],\left[\frac{f}{2}:1,1\right],\left[\frac{f+1}{2}:1,1\right]:[v_1+1:1],[v_2+1:1]:\end{matrix} ; -\frac{x^2}{4}, -\frac{x^2}{4}\right] \tag{2.7}$$

where

$a = \sigma + v_1 + v_2,\quad b = \sigma + \gamma - \alpha - \alpha' - \beta + v_1 + v_2,\quad c = \sigma + \beta' - \alpha' + v_1 + v_2,$

$d = \sigma + \gamma - \alpha - \alpha' + v_1 + v_2,\quad e = \sigma + \gamma - \alpha' - \beta + v_1 + v_2,\quad f = \sigma + \beta' + v_1 + v_2$ and

$F_{6:1,\ldots,1}^{6:0,\ldots,0}[\cdot]$ is defined in (1.18).

This corollary follows from Theorem 1, if we put $n = 2, a_1 = 1, a_2 = 1, \rho_1 = 1, \rho_2 = 1,$ by using (1.11) and take into account the relation

$$(z)_{2k} = 2^{2k}\left(\frac{z}{2}\right)_k\left(\frac{z+1}{2}\right)_k \qquad (z \in C, k \in N_0) \text{ and } N = (1, 2, \ldots, n) \tag{2.8}$$

where $(z)_k$ is the Pochhammer symbol (1.4).

Remark 1. When $n=1, a_1=1, \rho_1=1, v_1=v$ equation (2.4) is reduced to

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}J_v(t)\right]\right)(x)$$
$$=\frac{x^{\sigma+\beta'-\alpha+v-1}}{2^v}\frac{\Gamma(\sigma+v)\Gamma(\sigma+v-\alpha-\alpha'-\beta+\gamma)\Gamma(\sigma+v+\beta'-\alpha')}{\Gamma(\sigma+v+\gamma-\alpha-\alpha')\Gamma(\sigma+v+\gamma-\alpha'-\beta)\Gamma(\sigma+v+\beta'+\gamma)\Gamma(v+1)}$$

$$\times\,{}_6F_7\left[\begin{matrix}\frac{\sigma+v}{2},\frac{\sigma+v+1}{2},\frac{\sigma+v-\alpha-\alpha'-\beta+\gamma}{2},\frac{\sigma+v-\alpha-\alpha'-\beta+\gamma+1}{2},\frac{\sigma+v+\beta'-\alpha'}{2},\frac{\sigma+v+\beta'-\alpha'+1}{2}\\ \frac{\sigma+v+\gamma-\alpha-\alpha'}{2},\frac{\sigma+v+\gamma-\alpha-\alpha'+1}{2},\frac{\sigma+v+\gamma-\alpha'-\beta}{2},\frac{\sigma+v+\gamma-\alpha'-\beta+1}{2},\frac{\sigma+v+\beta'}{2},\frac{\sigma+v+\beta'+1}{2},v+1\end{matrix};-\frac{x^2}{4}\right] \quad (2.9)$$

3. Right-Sided Fractional Integration of Bessel Functions

The following result produce generalized right-hand sided fractional integration involving Appell function F_3 (1.2) of the product of Bessel functions.

Theorem 2.

Let $n\in N, \alpha,\alpha',\beta,\beta',\gamma,\sigma,v_j\in C$ and $a_j,\rho_j\in R_+\ (j=1,2,...,n)$ be such that

$$\mathrm{Re}(\gamma)>0,\ \mathrm{Re}(v_j)>-1,\ \mathrm{Re}\left(\sigma-\sum_{j=1}^{n}\rho_j v_j\right)<1+\min\left[\mathrm{Re}(-\beta),\mathrm{Re}(\alpha+\alpha'-\gamma),\mathrm{Re}(\alpha+\beta'-\gamma)\right] \quad (3.1)$$

then there holds the formula

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}\prod_{j=1}^{n}J_{v_j}\left(\frac{a_j}{t^{\rho_j}}\right)\right]\right)(x)=x^{\sigma-\alpha-\alpha'+\gamma-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{a_j}{2x^{\rho_j}}\right)^{v_j}}{\Gamma(v_j+1)}\right)\frac{\Gamma(g)\Gamma(h)\Gamma(l)}{\Gamma(u)\Gamma(v)\Gamma(w)}$$

$$\times F_{3:1,\dots,1}^{3:0,\dots,0}\left[\begin{matrix}[g:2\rho_1,...,2\rho_n],[h:2\rho_1,...,2\rho_n],[l:2\rho_1,...,2\rho_n]:\\ [u:2\rho_1,...,2\rho_n],[v:2\rho_1,...,2\rho_n],[w:2\rho_1,...,2\rho_n]:[v_1+1:1],...,[v_n+1:1]:\end{matrix};-\frac{a_1^2}{4x^{2\rho_1}},....,-\frac{a_n^2}{4x^{2\rho_n}}\right]; \quad (3.2)$$

where $g=1+\alpha+\alpha'-\gamma-\sigma+\sum_{j=1}^{n}(v_j\rho_j)$, $\quad h=1+\alpha+\beta'-\gamma-\sigma+\sum_{j=1}^{n}(v_j\rho_j)$,

$l=1-\beta-\sigma+\sum_{j=1}^{n}(v_j\rho_j)$, $\quad u=1-\sigma+\sum_{j=1}^{n}(v_j\rho_j)$,

$v=1+\alpha+\alpha'+\beta'-\gamma-\sigma+\sum_{j=1}^{n}(v_j\rho_j)$, $\quad w=1+\alpha-\beta-\sigma+\sum_{j=1}^{n}(v_j\rho_j)$; and $F_{3:1,\dots,1}^{3:0,\dots,0}[.]$ is given by (1.11)

Proof: First, we note that Δ_i in (1.14) is given by $\Delta_i = 1+n > 0 \quad (i=1,2,......,n \in N)$, and therefore $F^{3:0,....,0}_{3:1,....,1}[.]$ in the right hand side (3.2) is defined. Now, we prove (3.2). Applying equation (1.10) using (1.2) and changing the order of integration and summation, we get

$$\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{x,\infty}\left[t^{\sigma-1}\prod_{j=1}^{n} J_{v_j}\left(\frac{a_j}{t^{\rho_j}}\right)\right]\right)(x)$$

$$= I^{\alpha,\alpha',\beta,\beta',\gamma}_{x,\infty}\left[t^{\sigma-1}\left(\sum_{k_1=0}^{\infty}\frac{(-1)^{k_1}\left(\frac{a_1}{2t^{\rho_1}}\right)^{v_1+2k_1}}{\Gamma(v_1+k_1+1)\; k_1!}\right).......\left(\sum_{k_n=0}^{\infty}\frac{(-1)^{k_n}\left(\frac{a_n}{2t^{\rho_n}}\right)^{v_n+2k_n}}{\Gamma(v_n+k_n+1)\; k_n!}\right)\right](x)$$

$$= \sum_{k_1,....,k_n=0}^{\infty}\frac{(-1)^{k_1}\left(\frac{a_1}{2}\right)^{v_1+2k_1}}{\Gamma(v_1+1)\;(v_1+1)_{k_1}\; k_1!}........\frac{(-1)^{k_n}\left(\frac{a_n}{2}\right)^{v_n+2k_n}}{\Gamma(v_n+1)\;(v_n+1)_{k_n}\; k_n!}\times\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{x,\infty}\left\{t^{\sigma-v_1\rho_1-....-v_n\rho_n-2\rho_1k_1-....-2\rho_nk_n-1}\right\}\right)(x).$$

With the help of (3.1), for any $k_j \in N_0 \quad (j=1,2,.....,n)$

$\mathrm{Re}\left(\sigma-\sum_{j=1}^{n}v_j\rho_j-2\sum_{j=1}^{n}\rho_jk_j\right)\le \mathrm{Re}\left(\sigma-\sum_{j=1}^{n}v_j\rho_j\right)<1+\min\left[\mathrm{Re}(-\beta),\mathrm{Re}(\alpha+\alpha'-\gamma),\mathrm{Re}(\alpha+\beta'-\gamma)\right]$. Applying Lemma 1(ii) and using (2.2) with x and ρ are replaced by t and $\left(\sigma-\sum_{j=1}^{n}\rho_jv_j-2\sum_{j=1}^{n}\rho_jk_j\right)$ respectively for $j=1,2,......,n$. Then, we have

$$\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{x,\infty}\left[t^{\sigma-1}\prod_{j=1}^{n} J_{v_j}\left(\frac{a_j}{t^{\rho_j}}\right)\right]\right)(x)$$

$$= \sum_{k_1,....,k_n=0}^{\infty}\frac{(-1)^{k_1}\left(\frac{a_1}{2}\right)^{v_1+2k_1}}{\Gamma(v_1+1)\;(v_1+1)_{k_1}\; k_1!}........\frac{(-1)^{k_n}\left(\frac{a_n}{2}\right)^{v_n+2k_n}}{\Gamma(v_n+1)\;(v_n+1)_{k_n}\; k_n!}$$

$$\times\frac{\Gamma\left(1+\alpha+\alpha'-\gamma-\sigma+\sum_{j=1}^{n}(v_j\rho_j+2\rho_jk_j)\right)\;\Gamma\left(1+\alpha+\beta'-\gamma-\sigma+\sum_{j=1}^{n}(v_j\rho_j+2\rho_jk_j)\right)}{\Gamma\left(1-\sigma+\sum_{j=1}^{n}(v_j\rho_j+2\rho_jk_j)\right)\;\Gamma\left(1+\alpha+\alpha'+\beta'-\gamma-\sigma+\sum_{j=1}^{n}(v_j\rho_j+2\rho_jk_j)\right)}$$

$$\times\frac{\Gamma\left(1-\beta-\sigma+\sum_{j=1}^{n}(v_j\rho_j+2\rho_jk_j)\right)}{\Gamma\left(1+\alpha-\beta-\sigma+\sum_{j=1}^{n}(v_j\rho_j+2\rho_jk_j)\right)}\times x^{\sigma-\alpha-\alpha'+\gamma-1-\sum_{j=1}^{n}(v_j\rho_j+2\rho_jk_j)}$$

$$= x^{\sigma-\alpha-\alpha'+\gamma-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{a_j}{2x^{\rho_j}}\right)^{v_j}}{\Gamma(v_j+1)}\right)\frac{\Gamma\left(1+\alpha+\alpha'-\gamma-\sigma+\sum_{j=1}^{n}(v_j\rho_j)\right)\;\Gamma\left(1+\alpha+\beta'-\gamma-\sigma+\sum_{j=1}^{n}(v_j\rho_j)\right)}{\Gamma\left(1-\sigma+\sum_{j=1}^{n}(v_j\rho_j)\right)\;\Gamma\left(1+\alpha+\alpha'+\beta'-\gamma-\sigma+\sum_{j=1}^{n}(v_j\rho_j)\right)}$$

$$\times\frac{\Gamma\left(1-\beta-\sigma+\sum_{j=1}^{n}(v_j\rho_j)\right)}{\Gamma\left(1+\alpha-\beta-\sigma+\sum_{j=1}^{n}(v_j\rho_j)\right)}$$

$$\times\sum_{k_1,....,k_n=0}^{\infty}\frac{\left(1+\alpha+\alpha'-\gamma-\sigma+\sum_{j=1}^{n}v_j\rho_j\right)_{2\rho_1k_1+...+2\rho_nk_n}\left(1+\alpha+\beta'-\gamma-\sigma+\sum_{j=1}^{n}v_j\rho_j\right)_{2\rho_1k_1+...+2\rho_nk_n}}{\left(1-\sigma+\sum_{j=1}^{n}v_j\rho_j\right)_{2\rho_1k_1+...+2\rho_nk_n}\left(1+\alpha+\alpha'+\beta'-\gamma-\sigma+\sum_{j=1}^{n}v_j\rho_j\right)_{2\rho_1k_1+...+2\rho_nk_n}}$$

$$\times \frac{\Gamma\left(1-\beta-\sigma+\sum_{j=1}^{n} v_j\rho_j\right)_{2\rho_1 k_1+...+2\rho_n k_n}}{\Gamma\left(1+\alpha-\beta-\sigma+\sum_{j=1}^{n} v_j\rho_j\right)_{2\rho_1 k_1+...+2\rho_n k_n}} \times \frac{1}{(v_1+1)_{k_1}....(v_n+1)_{k_n}} \frac{\left(-\frac{a_1^2}{4x^{2\rho_1}}\right)^{k_1}}{k_1!} .... \frac{\left(-\frac{a_n^2}{4x^{2\rho_n}}\right)^{k_n}}{k_n!} .$$

(By using Pochhammer symbol $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ or $\Gamma(a+n) = \Gamma(a).(a)_n$).

In compliance with equation (1.11), we get the result in (3.2). This completes the proof of the theorem. Now, if we follow Theorem 2 in respective case $\alpha' = \beta' = 0, \ \ \beta = -\eta, \ \ \alpha = \alpha + \beta, \ \ \gamma = \alpha$. Then we have following result:

Corollary 2.1.

Let $n \in N, \ \alpha, \beta, \eta, \sigma, v_j \in C$ and $a_j, \rho_j \in R_+ \ (j = 1, 2, ..., n)$

be such that $\mathrm{Re}(\alpha) > 0, \ \mathrm{Re}(v_j) > -1, \ \ \mathrm{Re}\left(\sigma - \sum_{j=1}^{n} \rho_j v_j\right) < 1 + \min\left[\mathrm{Re}(\beta), \mathrm{Re}(\eta)\right]$.

Then, we have

$$\left(I_{x,\infty}^{\alpha,\beta,\eta}\left[t^{\sigma-1}\prod_{j=1}^{n} J_{v_j}\left(\frac{a_j}{t^{\rho_j}}\right)\right]\right)(x) = x^{\sigma-\beta-1}\left(\prod_{j=1}^{n} \frac{\left(\frac{a_j}{2x^{\rho_j}}\right)^{v_j}}{\Gamma(v_j+1)}\right) \frac{\Gamma(p)\Gamma(q)}{\Gamma(r)\Gamma(s)}$$

$$\times F_{2:1,....,1}^{2:0,...,0}\left[\begin{matrix} [p:2\rho_1,...,2\rho_n],[q:2\rho_1,...,2\rho_n]: \\ [r:2\rho_1,...,2\rho_n],[s:2\rho_1,...,2\rho_n]:[v_1+1:1],.......,[v_n+1:1]: \end{matrix} ; -\frac{a_1^2}{4x^{2\rho_1}}, ..., -\frac{a_n^2}{4x^{2\rho_n}}\right] \quad (3.3)$$

where

$p = 1 + \beta - \sigma + \sum_{j=1}^{n} \rho_j v_j \ , \quad q = 1 + \eta - \sigma + \sum_{j=1}^{n} \rho_j v_j \ ,$

$r = 1 - \sigma + \sum_{j=1}^{n} \rho_j v_j \ , \ \ s = 1 + \alpha + \beta + \eta - \sigma + \sum_{j=1}^{n} \rho_j v_j$, and $F_{2:1,....,1}^{2:0,...,0}[.]$ is given by (1.11).

If we put $\beta = -\alpha$ in Corollary 2.1 then we have following result:

Corollary 2.2.

Let $n \in N, \ \alpha, \sigma, v_j \in C$ and $a_j, \rho_j \in R_+ \ (j = 1, 2, ..., n)$

be such that $\mathrm{Re}(v_j) > -1 \ \ and \ \ 0 < \mathrm{Re}(\alpha) < 1 - \mathrm{Re}\left(\sigma - \sum_{j=1}^{n} \rho_j v_j\right)$. Then, we have

$$\left(I_{-}^{\alpha}\left[t^{\sigma-1}\prod_{j=1}^{n} J_{v_j}\left(\frac{a_j}{t^{\rho_j}}\right)\right]\right)(x) = x^{\sigma+\alpha-1}\left(\prod_{j=1}^{n} \frac{\left(\frac{a_j}{2x^{\rho_j}}\right)^{v_j}}{\Gamma(v_j+1)}\right) \frac{\Gamma\left(1-\sigma-\alpha+\sum_{j=1}^{n} \rho_j v_j\right)}{\Gamma\left(1-\sigma+\sum_{j=1}^{n} \rho_j v_j\right)}$$

$$\times F_{1:1,....,1}^{1:0,...,0}\left[\begin{matrix} \left[1-\sigma-\alpha+\sum_{j=1}^{n}\rho_j v_j:2\rho_1,...,2\rho_n\right]: \\ \left[1-\sigma+\sum_{j=1}^{n}\rho_j v_j:2\rho_1,...,2\rho_n\right]:[v_1+1:1],.......,[v_n+1:1]: \end{matrix} ; -\frac{a_1^2}{4x^{2\rho_1}}, ..., -\frac{a_n^2}{4x^{2\rho_n}}\right] \quad (3.4)$$

Corollary 2.3.

Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma,v_1,v_2\in C$ and $a_1,a_2,\rho_1,\rho_2\in R_+$ be such that

$\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(v_1)>-1,\mathrm{Re}(v_2)>-1,$

$\mathrm{Re}(\sigma-\rho_1 v_1-\rho_2 v_2)<1+\min\left[\mathrm{Re}(-\beta),\mathrm{Re}(\alpha+\alpha'-\gamma),\mathrm{Re}(\alpha+\beta'-\gamma)\right]$

and $\mathrm{Re}(\alpha+\alpha'-\gamma-\sigma+v_1+v_2+1)>0,\ \mathrm{Re}(\alpha+\beta'-\gamma-\sigma+v_1+v_2+1)>0$.

Then, we have

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}J_{v_1}\left(\frac{1}{t}\right)J_{v_2}\left(\frac{1}{t}\right)\right]\right)(x)=\frac{x^{-f}}{2^{v_1+v_2}}\frac{\Gamma(f)\Gamma(g)\Gamma(h)}{\Gamma(l)\Gamma(m)\Gamma(z)\Gamma(v_1+1)\Gamma(v_2+1)}$$

$$\times F_{6:1,\ldots,1}^{6:0,\ldots,0}\left[\begin{matrix}\left[\frac{f}{2}:1,1\right],\left[\frac{f+1}{2}:1,1\right],\left[\frac{g}{2}:1,1\right],\left[\frac{g+1}{2}:1,1\right],\left[\frac{h}{2}:1,1\right],\left[\frac{h+1}{2}:1,1\right]: \\ \left[\frac{l}{2}:1,1\right],\left[\frac{l+1}{2}:1,1\right],\left[\frac{m}{2}:1,1\right],\left[\frac{m+1}{2}:1,1\right],\left[\frac{z}{2}:1,1\right],\left[\frac{z+1}{2}:1,1\right]:[v_1+1:1],[v_2+1:1]:\end{matrix}\ ;-\frac{1}{4x^2},-\frac{1}{4x^2}\right], \tag{3.5}$$

where $f=1+\alpha+\alpha'-\gamma-\sigma+v_1+v_2$, $g=1+\alpha+\beta'-\gamma-\sigma+v_1+v_2$, $h=1-\beta-\sigma+v_1+v_2$, $l=1-\sigma+v_1+v_2$, $m=1+\alpha+\alpha'+\beta'-\gamma-\sigma+v_1+v_2$, $z=1+\alpha-\beta-\sigma+v_1+v_2$ and $F_{6:1,\ldots,1}^{6:0,\ldots,0}[\cdot]$ is defined in (1.18).

This corollary follows from Theorem 2, if we put $n=2,\ a_1=1,\ a_2=1,\ \rho_1=1,\ \rho_2=1,$ with using (1.11) and take into account the relation (2.8).

Remark 2. If we take $n=1,\ a_1=1,\ \rho_1=1,\ v_1=v$, equation (3.2) is reduced to

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}J_v\left(\frac{1}{t}\right)\right]\right)(x)$$

$$=\frac{x^{\sigma-\alpha-\alpha'-v+\gamma-1}}{2^v}\frac{\Gamma(1+\alpha+\alpha'-\gamma-\sigma+v)\Gamma(1+\alpha+\beta'-\gamma-\sigma+v)\Gamma(1-\beta-\sigma+v)}{\Gamma(1-\sigma+v)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\sigma+v)\Gamma(1+\alpha-\beta-\sigma+v)\Gamma(v+1)}$$

$$\times {}_6F_7\left[\begin{matrix}\frac{\alpha+\alpha'-\gamma-\sigma+v+1}{2},\frac{\alpha+\alpha'-\gamma-\sigma+v+2}{2},\frac{\alpha+\beta'-\gamma-\sigma+v+1}{2},\frac{\alpha+\beta'-\gamma-\sigma+v+2}{2},\frac{1-\beta-\sigma+v}{2},\frac{2-\beta-\sigma+v}{2} \\ v+1,\frac{1-\sigma+v}{2},\frac{2-\sigma+v}{2},\frac{\alpha+\alpha'+\beta'-\gamma-\sigma+v+1}{2},\frac{\alpha+\alpha'+\beta'-\gamma-\sigma+v+2}{2},\frac{\alpha-\beta-\sigma+v+1}{2},\frac{\alpha-\beta-\sigma+v+2}{2}\end{matrix};\frac{-1}{4x^2}\right]. \tag{3.6}$$

4. Fractional Integration of Cosine and Sine Functions

For $\nu = -\frac{1}{2}$ and $\nu = \frac{1}{2}$, the Bessel function $J_\nu(z)$ in (1.10) coincides with cosine- and sine-functions, apart from the multiplier $\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}$:

$$J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin(z), \qquad J_{-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos(z) \quad . \tag{4.1}$$

Theorem 3.

Let $\alpha, \alpha', \beta, \beta', \gamma, \sigma \in C$ and $a_j \in R_+ \ (j = 1, 2, ..., n)$, and setting

$\nu_1 = ... = \nu_n = -\frac{1}{2}$ and $\rho_1 = ... = \rho_n = 1$, be such that

$\mathrm{Re}(\gamma) > 0, \ \mathrm{Re}(\sigma) > 0,$

$\mathrm{Re}(\sigma + \gamma - \alpha - \alpha' - \beta) > 0, \ \ \mathrm{Re}(\sigma) > \max\left[0, \mathrm{Re}(\alpha + \alpha' + \beta' - \gamma), \mathrm{Re}(\alpha' - \beta')\right]$, from Theorem 1 we deduce the following result:

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}\prod_{j=1}^{n}\cos\left(a_j t\right)\right]\right)(x) = x^{\sigma-\alpha-\alpha'+\gamma-1}\,\frac{\Gamma(\sigma)\Gamma(\sigma+\gamma-\alpha-\alpha'-\beta)\Gamma(\sigma+\beta'-\alpha')}{\Gamma(\sigma+\gamma-\alpha-\alpha')\Gamma(\sigma+\gamma-\alpha'-\beta)\Gamma(\sigma+\beta')}$$

$$\times\, F_{3:1,....,1}^{3:0,...,0}\left[\begin{matrix}[\sigma:2,....,2],[\sigma+\gamma-\alpha-\alpha'-\beta:2,....,2],[\sigma+\beta'-\alpha':2,....,2]: \\ [\sigma+\gamma-\alpha-\alpha':2,....,2],[\sigma+\gamma-\alpha'-\beta:2,....,2],[\sigma+\beta':2,....,2]:\left[\frac{1}{2}:1\right],.....,\left[\frac{1}{2}:1\right]:\end{matrix}\ ;-\frac{a_1^2x^2}{4},.....,-\frac{a_n^2x^2}{4}\right]. \tag{4.2}$$

Corollary 3.1.

Let $\alpha, \beta, \eta, \sigma \in C$ and $a_j \in R_+ \ \ (j = 1, 2, ..., n)$

be such that $\mathrm{Re}(\sigma) > 0, \ \mathrm{Re}(\alpha) > 0, \ \mathrm{Re}(\sigma + \eta - \beta) > 0, \ \ \mathrm{Re}(\sigma) > \max\left[0, \mathrm{Re}(\beta - \eta)\right]$ with setting

$\nu_1 = ... = \nu_n = -\frac{1}{2}$ and $\rho_1 = ... = \rho_n = 1$, from Corollary 1.1, we hold

$$\left(I_{0,x}^{\alpha,\beta,\eta}\left[t^{\sigma-1}\prod_{j=1}^{n}\cos\left(a_j t\right)\right]\right)(x) = x^{\sigma-\beta-1}\,\frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)}$$

$$\times\, F_{2:1,....,1}^{2:0,...,0}\left[\begin{matrix}[\sigma:2,....,2],[\sigma+\eta-\beta:2,....,2]: \\ [\sigma-\beta:2,....,2],[\sigma+\alpha+\eta:2,....,2]:\left[\frac{1}{2}:1\right],.....,\left[\frac{1}{2}:1\right]:\end{matrix}\ ;-\frac{a_1^2x^2}{4},.....,-\frac{a_n^2x^2}{4}\right]. \tag{4.3}$$

Corollary 3.2.

Let $\alpha, \sigma \in C$ and $a_j \in R_+ \ (j = 1, 2, ..., n)$ be such that $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\sigma) > 0$ with setting

$\nu_1 = ... = \nu_n = -\frac{1}{2}$ and $\rho_1 = ... = \rho_n = 1$,

from Corollary 1.2, we have

$$\left({}_0I_x^{\alpha}\left[t^{\sigma-1}\prod_{j=1}^{n}\cos\left(a_j t\right)\right]\right)(x) = x^{\sigma+\alpha-1}\frac{\Gamma(\sigma)}{\Gamma(\sigma+\alpha)}$$

$$\times F_{1:1,\ldots,1}^{1:0,\ldots,0}\left[\begin{matrix}[\sigma:2,\ldots,2]: \\ [\sigma+\alpha:2,\ldots,2]:\left[\frac{1}{2}:1\right],\ldots,\left[\frac{1}{2}:1\right]:\end{matrix}\;;-\frac{a_1^2x^2}{4},\ldots,-\frac{a_n^2x^2}{4}\right] \tag{4.4}$$

Theorem 4.

Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma\in C$ and $a_j\in R_+\ (j=1,2,\ldots,n)$, and setting

$v_1=\ldots\ldots=v_n=\frac{1}{2}$ and $\rho_1=\ldots\ldots=\rho_n=1$ be such that

$\mathrm{Re}(\gamma)>0,\quad \mathrm{Re}(\sigma)>0,$

$\mathrm{Re}(\sigma+\gamma-\alpha-\alpha'-\beta)>0,\quad \mathrm{Re}(\sigma)>\max\left[0,\mathrm{Re}(\alpha+\alpha'+\beta'-\gamma),\mathrm{Re}(\alpha'-\beta')\right]$, from

Theorem 1 we deduce the following result:

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-n-1}\prod_{j=1}^{n}\sin\left(a_j t\right)\right]\right)(x)=\frac{\pi^{\frac{n}{2}}}{2^n}\left(\prod_{j=1}^{n}a_j\right)x^{\sigma-\alpha-\alpha'+\gamma-1}$$

$$\times\frac{\Gamma(\sigma)\Gamma(\sigma+\gamma-\alpha-\alpha'-\beta)\Gamma(\sigma+\beta'-\alpha')}{\Gamma(\sigma+\gamma-\alpha-\alpha')\Gamma(\sigma+\gamma-\alpha'-\beta)\Gamma(\sigma+\beta')}$$

$$\times F_{3:1,\ldots,1}^{3:0,\ldots,0}\left[\begin{matrix}[\sigma:2,\ldots,2],[\sigma+\gamma-\alpha-\alpha'-\beta:2,\ldots,2],[\sigma+\beta'-\alpha':2,\ldots,2]: \\ [\sigma+\gamma-\alpha-\alpha':2,\ldots,2],[\sigma+\gamma-\alpha'-\beta:2,\ldots,2],[\sigma+\beta':2,\ldots,2]:\left[\frac{3}{2}:1\right],\ldots,\left[\frac{3}{2}:1\right]:\end{matrix}\;;-\frac{a_1^2x^2}{4},\ldots,-\frac{a_n^2x^2}{4}\right]. \tag{4.5}$$

Corollary 4.1.

Let $\alpha,\beta,\eta,\sigma\in C$ and $a_j\in R_+\ (j=1,2,\ldots,n)$

be such that $\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\sigma)>0,\ \ \mathrm{Re}(\sigma+\eta-\beta)>0,\ \ \mathrm{Re}(\sigma)>\max\left[0,\mathrm{Re}(\beta-\eta)\right]$ with setting

$v_1=\ldots\ldots=v_n=\frac{1}{2}$ and $\rho_1=\ldots\ldots=\rho_n=1$, from Corollary 1.1, we hold

$$\left(I_{0,x}^{\alpha,\beta,\eta}\left[t^{\sigma-n-1}\prod_{j=1}^{n}\sin\left(a_j t\right)\right]\right)(x)=\frac{\pi^{\frac{n}{2}}}{2^n}\left(\prod_{j=1}^{n}a_j\right)x^{\sigma-\beta-1}\frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)}$$

$$\times F_{2:1,\ldots,1}^{2:0,\ldots,0}\left[\begin{matrix}[\sigma:2,\ldots,2],[\sigma+\eta-\beta:2,\ldots,2]: \\ [\sigma-\beta:2,\ldots,2],[\sigma+\alpha+\eta:2,\ldots,2]:\left[\frac{3}{2}:1\right],\ldots,\left[\frac{3}{2}:1\right]:\end{matrix}\;;-\frac{a_1^2x^2}{4},\ldots,-\frac{a_n^2x^2}{4}\right]. \tag{4.6}$$

Corollary 4.2.

Let $\alpha,\sigma\in C$ and $a_j\in R_+\ (j=1,2,\ldots,n)$ be such that $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\sigma)>0$ with setting

$v_1=\ldots\ldots=v_n=\frac{1}{2}$ and $\rho_1=\ldots\ldots=\rho_n=1$, from Corollary 1.2, we have

$$\left({}_0I_x^{\alpha}\left[t^{\sigma-n-1}\prod_{j=1}^{n}\sin\left(a_jt\right)\right]\right)(x)=\frac{\pi^{\frac{n}{2}}}{2^n}\left(\prod_{j=1}^{n}a_j\right)x^{\sigma-\beta-1}\frac{\Gamma(\sigma)}{\Gamma(\sigma+\alpha)}$$

$$\times F_{1:1,\ldots,1}^{1:0,\ldots,0}\left[\begin{matrix}[\sigma:2,\ldots,2]: \\ [\sigma+\alpha:2,\ldots,2]:\left[\frac{3}{2}:1\right],\ldots,\left[\frac{3}{2}:1\right]:\end{matrix};-\frac{a_1^2x^2}{4},\ldots,-\frac{a_n^2x^2}{4}\right] \quad (4.7)$$

Similarly, setting $v_1=\ldots\ldots=v_n=-\frac{1}{2}$, $v_1=\ldots\ldots=v_n=\frac{1}{2}$ and $\rho_1=\ldots\ldots=\rho_n=1$, and taking (4.1) into account, from Theorem 2 and Corollaries 2.1 and 2.2, we get the following results:

Theorem 5.

Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma\in C$ and $a_j\in R_+\ (j=1,2,\ldots,n)$ be such that $\operatorname{Re}(\gamma)>0,\ \operatorname{Re}(\alpha+\alpha'-\gamma-\sigma)>0,\ \operatorname{Re}(\alpha+\beta'-\gamma-\sigma)>0,\ \operatorname{Re}(-\sigma-\beta)>0$ and $\operatorname{Re}(\sigma)<\min\left[\operatorname{Re}(-\beta),\operatorname{Re}(\alpha+\alpha'-\gamma),\operatorname{Re}(\alpha+\beta'-\gamma)\right]$.

Then, we hold the formula

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma}\prod_{j=1}^{n}\cos\left(\frac{a_j}{t}\right)\right]\right)(x)=x^{\sigma-\alpha-\alpha'+\gamma}\frac{\Gamma(\alpha+\alpha'-\gamma-\sigma)\Gamma(\alpha+\beta'-\gamma-\sigma)\Gamma(-\sigma-\beta)}{\Gamma(-\sigma)\Gamma(\alpha+\alpha'+\beta'-\gamma-\sigma)\Gamma(\alpha-\beta-\sigma)}$$

$$\times F_{3:1,\ldots,1}^{3:0,\ldots,0}\left[\begin{matrix}[\alpha+\alpha'-\gamma-\sigma:2,\ldots,2],[\alpha+\beta'-\gamma-\sigma:2,\ldots,2],[-\sigma-\beta:2,\ldots,2]: \\ [-\sigma:2,\ldots,2],[\alpha+\alpha'+\beta'-\gamma-\sigma:2,\ldots,2],[\alpha-\beta-\sigma:2,\ldots,2]:\left[\frac{1}{2}:1\right],\ldots,\left[\frac{1}{2}:1\right]:\end{matrix};-\frac{a_1^2}{4x^2},\ldots,-\frac{a_n^2}{4x^2}\right]. \quad (4.8)$$

The above result obtained from Theorem 2 by sifting $v_1=\ldots\ldots=v_n=-\frac{1}{2}$ and $\rho_1=\ldots\ldots=\rho_n=1$ with using (4.1).

Corollary 5.1.

Let $\alpha,\beta,\eta,\sigma\in C$ and $a_j\in R_+\ (j=1,2,\ldots,n)$ be such that $\operatorname{Re}(\alpha)>0,\ \operatorname{Re}(\beta-\sigma)>0,\ \operatorname{Re}(\eta-\sigma)>0,\ \operatorname{Re}(\sigma)<\min\left[\operatorname{Re}(\beta),\operatorname{Re}(\eta)\right]$.Then we get,

$$\left(I_{x,\infty}^{\alpha,\beta,\eta}\left[t^{\sigma}\prod_{j=1}^{n}\cos\left(\frac{a_j}{t}\right)\right]\right)(x)=x^{\sigma-\beta}\frac{\Gamma(\beta-\sigma)\Gamma(\eta-\sigma)}{\Gamma(-\sigma)\Gamma(\alpha+\beta+\eta-\sigma)}$$

$$\times F_{2:1,\ldots,1}^{2:0,\ldots,0}\left[\begin{matrix}[\beta-\sigma:2,\ldots,2],[\eta-\sigma:2,\ldots,2]: \\ [-\sigma:2,\ldots,2],[\alpha+\beta+\eta-\sigma:2,\ldots,2]:\left[\frac{1}{2}:1\right],\ldots,\left[\frac{1}{2}:1\right]:\end{matrix};-\frac{a_1^2}{4x^2},\ldots,-\frac{a_n^2}{4x^2}\right]. \quad (4.9)$$

Corollary 5.2.

Let $\alpha,\sigma\in C$ and $a_j\in R_+\ (j=1,2,\ldots,n)$ be such that $0<\operatorname{Re}(\alpha)<-\operatorname{Re}(\sigma)$.Then we have

$$\left(I_{-}^{\alpha}\left[t^{\sigma}\prod_{j=1}^{n}\cos\left(\frac{a_j}{t}\right)\right]\right)(x)=x^{\sigma+\alpha}\frac{\Gamma(-\alpha-\sigma)}{\Gamma(-\sigma)}$$

$$\times F_{1:1,\ldots,1}^{1:0,\ldots,0}\left[\begin{matrix}[-\alpha-\sigma:2,\ldots,2]: \\ [-\sigma:2,\ldots,2]:\left[\frac{1}{2}:1\right],\ldots,\left[\frac{1}{2}:1\right]:\end{matrix};-\frac{a_1^2}{4x^2},\ldots,-\frac{a_n^2}{4x^2}\right]. \tag{4.10}$$

The Corollaries 5.1 and 5.2 obtained from Corollaries 2.1 and 2.2 by setting

$v_1=\ldots\ldots=v_n=-\frac{1}{2}$ and $\rho_1=\ldots\ldots=\rho_n=1$ with using (4.1).

Theorem 6.

Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma\in C$ and $a_j\in R_+\ (j=1,2,\ldots,n)$ be such that

$\mathrm{Re}(\gamma)>0,\ \ \mathrm{Re}(\alpha+\alpha'-\gamma-\sigma)>-1,\ \ \mathrm{Re}(\alpha+\beta'-\gamma-\sigma)>-1,\ \mathrm{Re}(-\sigma-\beta)>-1$ and

$\mathrm{Re}(\sigma)<1+\min\left[\ \mathrm{Re}(-\beta),\ \mathrm{Re}(\alpha+\alpha'-\gamma),\ \mathrm{Re}(\alpha+\beta'-\gamma)\right]$. Then, we hold the formula

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma+n-1}\prod_{j=1}^{n}\sin\left(\frac{a_j}{t}\right)\right]\right)(x)=\frac{\pi^{\frac{n}{2}}}{2^n}\left(\prod_{j=1}^{n}a_j\right)x^{\sigma-\alpha-\alpha'+\gamma-1}$$

$$\times\frac{\Gamma(\alpha+\alpha'-\gamma-\sigma+1)\Gamma(\alpha+\beta'-\gamma-\sigma+1)\Gamma(1-\sigma-\beta)}{\Gamma(1-\sigma)\Gamma(\alpha+\alpha'+\beta'-\gamma-\sigma+1)\Gamma(\alpha-\beta-\sigma+1)}$$

$$\times F_{3:1,\ldots,1}^{3:0,\ldots,0}\left[\begin{matrix}[\alpha+\alpha'-\gamma-\sigma+1:2,\ldots,2],[\alpha+\beta'-\gamma-\sigma+1:2,\ldots,2],[1-\sigma-\beta:2,\ldots,2]: \\ [1-\sigma:2,\ldots,2],[\alpha+\alpha'+\beta'-\gamma-\sigma+1:2,\ldots,2],[\alpha-\beta-\sigma+1:2,\ldots,2]:\left[\frac{3}{2}:1\right],\ldots,\left[\frac{3}{2}:1\right]:\end{matrix};-\frac{a_1^2}{4x^2},\ldots,-\frac{a_n^2}{4x^2}\right]. \tag{4.11}$$

The above result obtained from Theorem 2 by setting

$v_1=\ldots\ldots=v_n=\frac{1}{2}$ and $\rho_1=\ldots\ldots=\rho_n=1$ with taking (4.1) into account.

Corollary 6.1.

Let $\alpha,\beta,\eta,\sigma\in C$ and $a_j\in R_+\ (j=1,2,\ldots,n)$

be such that $\mathrm{Re}(\alpha)>0,\ \ \mathrm{Re}(\beta-\alpha)>-1,\ \ \mathrm{Re}(\eta-\sigma)>-1,\ \ \mathrm{Re}(\sigma)<1+\min\left[\mathrm{Re}(\beta),\mathrm{Re}(\eta)\right]$.

Then, we have

$$\left(I_{x,\infty}^{\alpha,\beta,\eta}\left[t^{\sigma+n-1}\prod_{j=1}^{n}\sin\left(\frac{a_j}{t}\right)\right]\right)(x)=\frac{\pi^{\frac{n}{2}}}{2^n}\left(\prod_{j=1}^{n}a_j\right)x^{\sigma-\beta-1}\frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\alpha+\beta+\eta-\sigma+1)}$$

$$\times F_{2:1,\ldots,1}^{2:0,\ldots,0}\left[\begin{matrix}[\beta-\sigma+1:2,\ldots,2],[\eta-\sigma+1:2,\ldots,2]: \\ [1-\sigma:2,\ldots,2],[\alpha+\beta+\eta-\sigma+1:2,\ldots,2]:\left[\frac{3}{2}:1\right],\ldots,\left[\frac{3}{2}:1\right]:\end{matrix};-\frac{a_1^2}{4x^2},\ldots,-\frac{a_n^2}{4x^2}\right]. \tag{4.12}$$

Corollary 6.2.

Let $\alpha,\sigma\in C$ and $a_j\in R_+\ (j=1,2,\ldots,n)$ be such that $0<\mathrm{Re}(\alpha)<1-\mathrm{Re}(\sigma)$. Then, we have

$$\left(I_{-}^{\alpha}\left[t^{\sigma+n-1}\prod_{j=1}^{n}\sin\left(\frac{a_j}{t}\right)\right]\right)(x)=\frac{\pi^{\frac{n}{2}}}{2^n}\left(\prod_{j=1}^{n}a_j\right)x^{\sigma+\alpha-1}\frac{\Gamma(1-\alpha-\sigma)}{\Gamma(1-\sigma)}$$

$$\times F_{1:1,\ldots,1}^{1:0,\ldots,0}\left[\begin{matrix}[1-\alpha-\sigma:2,\ldots,2]: \\ [1-\sigma:2,\ldots,2]:\left[\frac{3}{2}:1\right],\ldots,\left[\frac{3}{2}:1\right]:\end{matrix};-\frac{a_1^2}{4x^2},\ldots,-\frac{a_n^2}{4x^2}\right] \tag{4.13}$$

The Corollaries 6.1 and 6.2 obtained from Corollaries 2.1 and 2.2 by sifting

$v_1=\ldots\ldots=v_n=\frac{1}{2}$ and $\rho_1=\ldots\ldots=\rho_n=1$ with taking (4.1) into account.

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Ratio Type Estimator of Population Mean in Double Sampling

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Abstract This paper discusses the problem of estimation of population mean in double sampling. In fact, in this paper double sampling version of (Singh and Tailor, 2003) has been suggested. The bias and mean squared error of the suggested estimators are obtained up to the first degree of approximation. The suggested estimator has been compared with simple mean estimator and double sampling ratio estimator. An empirical study has been carried out to demonstrate the performance of the suggested estimator.
Keywords *Double Sampling, Population, Ratio Type Estimator*

1. Introduction

It is a well known fact that the auxiliary information can increase the efficiency of the estimators of parameters of importance in survey sampling. It is shown that the estimator due to (Kawathekar and Ajgaonkar, 1984) is a member of the proposed class of estimators. Various authors (Kadilar and Cingi, 2004; Jhajj *et al*., 2006; Sodipo and Obisesan, 2007) have defined various estimators including ratio, regression, difference etc. for estimating the unknown population parameters of study variable y, by using the prior knowledge of population mean X of auxiliary variable x, which is highly correlated with study variable y. If the information on population mean X is missing then two phase (double) sampling technique has been generally recommended. In the two-phase sampling scheme, a large preliminary random sample (called first phase sample) is drawn from the population and information on auxiliary variable is taken, which is used to estimate the value of unknown population mean X of auxiliary variable x. Then second phase sample is drawn either from the first phase sample or independently from the population and observations on both study and auxiliary variable are taken. (Housila, 2012) estimated finite population mean in two-phase sampling with known coefficient of variation of an auxiliary character.

2. Materials and Methods

In this paper double sampling version of (Singh and Tailor, 2003) has been suggested.

3. Results and Discussion

Usual procedure of double sampling is described as below:

(i) A large sample S_1 of size $n'(n' < N)$ is drawn and observations are taken only on auxiliary variate to estimate population mean of auxiliary variate;

(ii) Then a sample S_2 of size $n(n < n')$ is drawn either from S_1 (case I) or directly from the population of size N to observe both study variate as well as auxiliary variate.

Let us consider a finite population $U = (U_1, U_2, ..., U_N)$ of size N. Let *y* and *x* be the study and auxiliary variates and y_1 and x_1 be the observations taken on study variate *y* and auxiliary variate *x* respectively. A sample of size *n* is drawn to estimation of the population mean $\hat{\bar{Y}}$ of the study variate *y*.

The classical ratio estimator given by (Cochran, 1940) is defined as,

$$\hat{\bar{Y}}_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) \quad (1.1)$$

Where, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ are unbiased estimators of population mean $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ and $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$ respectively.

When population mean of auxiliary variate $\bar{X}$ is not known, double sampling ratio estimator is defined as,

$$\hat{\bar{Y}}_R^d = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right) \quad (1.2)$$

Where, $\bar{x}' = \frac{1}{n'}\sum_{i=1}^{n'} x_i$ is an unbiased estimator of population mean $\bar{X}$ based on sample of size n'

Thus the bias and mean squared error of $\hat{\bar{Y}}_R^d$ up to the first degree of approximation is obtained as

$$B(\hat{\bar{Y}}_R^d) = \bar{Y} f_3 [C_x^2 (1 - K_{yx})] \quad (1.3)$$

$$MSE(\hat{\bar{Y}}_R^d) = \bar{Y}^2 [f_1 C_y^2 + C_x^2 f_3 (1 - 2\rho_{yx} C_y / C_x)]$$

$$MSE(\hat{\bar{Y}}_R^d) = \bar{Y}^2 [f_1 C_y^2 + C_x^2 f_3 (1 - 2K_{yx})] \quad (1.4)$$

2. Suggested Estimator

(Singh and Tailor, 2003) utilized information on correlation coefficient ρ_{xy} between study variate *y* and auxiliary variate *x* and suggested ratio type estimator as,

$$\hat{\bar{Y}}_{ST} = \bar{y}\left(\frac{\bar{X}+\rho_{yx}}{\bar{x}+\rho_{yx}}\right) \tag{2.1}$$

$\hat{\bar{Y}}_{ST}$ in double sampling is expressed as,

$$\hat{\bar{Y}}_{ST}^{(d)} = \bar{y}\left(\frac{\bar{x}'+\rho_{yx}}{\bar{x}+\rho_{yx}}\right) \tag{2.2}$$

To obtain the bias and mean squared error of the suggested estimator $\hat{\bar{Y}}_{ST}^{(d)}$ we write

$\bar{y} = \bar{Y}(1+e_0), \bar{x} = \bar{X}(1+e_1)$ and $\bar{x}' = \bar{X}(1+e_2)$

Such that $E(e_0) = E(e_1) = E(e_2) = 0$

$E(e_0^2) = f_1 C_y^2$, $E(e_1^2) = f_1 C_x^2$, $E(e_2^2) = f_2 C_x^2$, $E(e_0 e_1) = f_1 \rho_{yx} C_y C_x$

$E(e_0 e_2) = f_2 \rho_{yx} C_y C_x$, $E(e_1 e_2) = f_2 C_x^2$.

Expressing $\hat{\bar{Y}}_{ST}^{(d)}$ in terms of e_i' s we have

$$\hat{\bar{Y}}_{ST}^{(d)} = \bar{Y}(1+e_0)(1+\theta e_2)(1+\theta e_1)^{-1}$$

Thus, bias and mean squared error of $\hat{\bar{Y}}_{ST}^{(d)}$ upto first degree of approximation are obtained as,

$$B(\hat{\bar{Y}}_{ST}^{(d)}) = \bar{Y} f_3 \theta C_x^2 (\theta - K_{yx}) \tag{2.3}$$

$$MSE(\hat{\bar{Y}}_{ST}^{(d)}) = \bar{Y}^2 \left(f_1 C_y^2 + f_3 \theta C_x^2 (\theta - 2\rho_{yx} \frac{C_y}{C_x}) \right) \tag{2.4}$$

$$MSE(\hat{\bar{Y}}_{ST}^{(d)}) = \bar{Y}^2 \left[f_1 C_y^2 + f_3 \theta C_x^2 (\theta - 2K_{yx}) \right] \tag{2.5}$$

3. Efficiency Comparisons

It is well known under simple random sampling without replacement (SRSWOR) that

$$V(\bar{y}) = f_1 \bar{Y}^2 C_y^2 \tag{3.1}$$

Comparison of (2.4) and (3.1) shows that the suggested estimator $\hat{\bar{Y}}_{ST}^{(d)}$ would be more efficient than simple mean estimator $\bar{y}$ i.e.

$$MSE(\hat{\bar{Y}}_{ST}^{(d)}) - V(\bar{y}) < 0 \quad if \;\; 0 < \theta < 2K_{yx} \tag{3.2}$$

Comparing of (1.4) and (2.4), it is observed that the suggested estimator $\hat{\bar{Y}}_{ST}^{(d)}$ would more efficient than ratio estimator in double sampling $\hat{\bar{Y}}_{R}^{(d)}$ i.e.

$$MSE(\hat{\bar{Y}}_{ST}^{(d)}) - MSE(\hat{\bar{Y}}_{R}^{d}) < 0 \;\; if \;\; \left.\begin{array}{ll} either & 2k-1 < \theta < 1 \\ or & 1 < \theta < 2k-1 \end{array}\right\} \tag{3.3}$$

4. Empirical Study

To see the performance of the suggested estimator $\hat{\bar{Y}}_{ST}^{d}$ over simple mean estimator $\bar{y}$ and double sampling ratio estimator $\hat{\bar{Y}}_{R}^{d}$ two natural population data sets are being considered. Descriptions of the population are given below:

Population I: [**Source:** (Das, 1988)]

X: the number of agricultural laboures for 1961,
Y: the number of agricultural laboures for 1971,

$\bar{X}$ = 25.1110, $\bar{Y}$ = 39.0680, N = 278, n = 60, n' = 180
C_y= 1.4451, C_x = 1.6198, ρ_{yx} = 0.7213

Population II: [**Source:** (Cochran, 1977)]

x: The number of rooms per block
y: The number of persons per block

$\bar{Y}$ = 101.1, $\bar{X}$ = 58.80, C_y = 0.14450
C_x = 0.1281, ρ_{xy} = 0.6500, N = 20, n = 8, n' = 12

Table 1: *Percent Relative Efficiency of* $\bar{y}$, $\hat{\bar{Y}}_R^{(d)}$, $\hat{\bar{Y}}_{ST}^{(d)}$ *with Respect* $\bar{y}$

Estimator	PRE $(.,\bar{y})$	
	Population	
	I	II
$\bar{y}$	100	100
$\hat{\bar{Y}}_R^{(d)}$	142.11	117.65
$\hat{\bar{Y}}_{ST}^{(d)}$	150	125

Table 1 exhibit that the suggested ratio estimator $\hat{\bar{Y}}_{ST}^{(d)}$ has highest percent relative efficiency in comparison to simple mean estimator $\bar{y}$ and double sampling ratio estimator $\hat{\bar{Y}}_R^{(d)}$. Thus suggested estimator $\hat{\bar{Y}}_{ST}^{(d)}$ is recommended for its use in practice for the estimation of population mean.

In order to improve the efficiency of the estimators, auxiliary information is used at both selections as well as estimation stages to improve the efficiency of the estimators. (Cochran, 1940) used auxiliary information at estimation stage and proposed ratio estimator. (Murthy, 1964) envisaged product estimator, (Searls, 1964) used coefficient of variation of study variate, motivated by (Searls, 1964), (Sisodia and Dwivedi, 1981) utilized coefficient of variation of auxiliary variate. (Srivenkataramana, 1980) first proposed dual to ratio estimator. (Singh and Tailor, 2005) and (Tailor and Sharma, 2009) worked on ratio-cum-product estimators. These motivates author to propose a new ratio-cum–dual to ratio estimator utilizing dual to ratio estimator of finite population mean.

5. Conclusion

In this paper, the bias and mean squared error of the suggested estimators are obtained up to the first degree of approximation. The suggested estimator has been compared with simple mean estimator and double sampling ratio estimator. An empirical study has been carried out to demonstrate the performance of the suggested estimator.

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Research Article **Open Access**

An Approach to Construct a Control Chart for Standard Deviation Based on Six Sigma

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Abstract In present scenario companies started applying six sigma concept in their manufacturing process, which results in lesser number of defects compared with existing Shewhart control chart. In this article an attempt is made to construct a control chart based on standard deviation with six sigma especially planned for companies applying this technique in their business and constructed table also presented for the experts to take quick decisions.
Keywords *Control Chart; Process Control; Six Sigma; Six Sigma Quality Level*

1. Introduction

The control charts suggested by W.A. Shewhart (1931) was based on 3 sigma control limits. Radhakrishnan (2009a, 2009b, 200c, 2010a, 2010b, 2010c, 2011 and 2012) described as if the same charts are used for the products of the companies which adopt Six sigma concept in their process, and then no point will fall outside the control limits because of the improvement in the quality. So a separate control chart is necessary to check the results of the companies, which adopt Six sigma. The practice was aimed at taking action to improve the overall performance and the companies, which are practicing Six sigma, are expected to produce 3.4 or less number of defects per million opportunities. In this research an effort is made to construct a control chart based on the concept Six sigma for Standard deviation and the Table 1 is also provided for the experts to build rapid judgments on the floor itself.

2. Concepts and Terminologies

Upper Specification Limit (USL)

It is the greatest amount specified by the producer for a process or product to have the acceptable performance.

Lower Specification Limit (LSL)

It is the smallest amount specified by the producer for a process or product to have the acceptable performance.

Tolerance Level (TL)

It is the difference between USL and LSL, TL = USL-LSL

Process Capability (Cp)

This is the ratio of tolerance level to six times standard deviation of the process.

$$c_p = (TL / 6\sigma) = (USL - LSL / 6\sigma)$$

Standard Deviation (σ)

For many purposes standard deviation is the most useful measure of dispersion of a set of numbers. It is the root mean square value.

Quality Control Constant[1] ($O_{6\sigma}$)

The constant $O_{6\sigma}$ introduced in this paper to determine the control limits based on six sigma initiatives for standard deviation.

Quality Control Constant[2] (c_4)

The constant c_4 introduced in this paper to determine the control limits based on 3sigma for standard deviation.

3. Construction of Control Chart Based on Six Sigma Initiatives for Standard Deviation

Fix the tolerance level (TL) and process capability (C_P) to determine the process standard deviation ($\sigma_{6\sigma}$). Apply the value of $\sigma_{6\sigma}$ in the control limits $\bar{S} \pm \left\{ \left(O_{6\sigma} \times \sqrt{1 - c_4^{\,2}} \right) \times \sigma_{6\sigma} \right\}$, to get the control limits based on six sigma initiatives for Standard deviation. The value of $O_{6\sigma}$ is obtained using $p(z \le z_{6\sigma}) = 1 - \alpha_1, \alpha_1 = 3.4 \text{ x } 10^{-6}$ and z is a standard normal variate. For a specified TL and C_P of the process, the value of σ (termed as $\sigma_{6\sigma}$) is calculated from $c_p = \frac{TL}{6\sigma}$ using a C program and presented in Table 1.2 for various combinations of TL and C_P. The control limits based on six sigma initiatives for Standard deviation chart are

$$UCL_{6\sigma} = \bar{S} + \left\{ \left(O_{6\sigma} \times \sqrt{1 - c_4^{\,2}} \right) \right\} \times \sigma_{6\sigma}$$

$$Center\ line\ CL_{6\sigma} = \bar{S}$$

$$LCL_{6\sigma} = \bar{S} - \left\{ \left(O_{6\sigma} \times \sqrt{1 - c_4^{\,2}} \right) \right\} \times \sigma_{6\sigma}$$

Table 1: *Values for a specified Cp and TL*

C_p \ TL	0.0131	0.0132	0.0133	0.0134	0.0135
1.0	0.0022	0.0022	0.0022	0.0022	0.0023
1.1	0.0020	0.0020	0.0020	0.0020	0.0020
1.2	0.0018	0.0018	0.0018	0.0019	0.0019
1.3	0.0017	0.0017	0.0017	0.0017	0.0017
1.4	0.0016	0.0016	0.0016	0.0016	0.0016
1.5	0.0015	0.0015	0.0015	0.0015	0.0015
1.6	0.0014	0.0014	0.0014	0.0014	0.0014
1.7	0.0013	0.0013	0.0013	0.0013	0.0013
1.8	0.0012	0.0012	0.0012	0.0012	0.0013
1.9	0.0011	0.0012	0.0012	0.0012	0.0012
2.0	0.0011	0.0011	0.0011	0.0011	0.0011
2.1	0.0010	0.0010	0.0011	0.0011	0.0011
2.2	0.0010	0.0010	0.0010	0.0010	0.0010
2.3	0.0010	0.0010	0.0010	0.0010	0.0010
2.4	0.0009	0.0009	0.0009	0.0093	0.0009
2.5	0.0009	0.0009	0.0090	0.0089	0.0009

4. Conditions for Application

- Human involvement should be less in the manufacturing process
- The company adopts Six sigma quality initiatives in its processes

Example

The following results of inside Diameter Measurement (mm) for Automobile Engine Piston Rings.

Table 2: *Inside Diameter Measurement (mm) for*

Automobile Engine Piston Rings and Standard deviation (S_i *)*

Sample Number	Observation					S_i
1	74.030	74.002	74.019	73.992	74.008	0.0148
2	73.995	73.992	74.001	74.011	74.004	0.0075
3	73.988	74.024	74.021	74.005	74.002	0.0147
4	74.002	73.996	73.993	74.015	74.009	0.0091
5	73.992	74.007	74.015	73.989	74.014	0.0122
6	74.009	73.994	73.997	73.985	73.993	0.0087
7	73.995	74.006	73.994	74.000	74.005	0.0055
8	73.985	74.003	73.993	74.015	73.988	0.0123
9	74.008	73.995	74.009	74.005	74.004	0.0055
10	73.998	74.000	73.990	74.007	73.995	0.0063
11	73.994	73.998	73.994	73.995	73.990	0.0029
12	74.004	74.000	74.007	74.000	73.996	0.0042
13	73.983	74.002	73.998	73.997	74.012	0.0105
14	74.006	73.967	73.994	74.000	73.984	0.0153
15	74.012	74.014	73.998	73.999	74.007	0.0073
16	74.000	73.984	74.005	73.998	73.996	0.0078
17	73.994	74.012	73.986	74.005	74.007	0.0106
18	74.006	74.010	74.018	74.003	74.000	0.0070
19	73.984	74.002	74.003	74.005	73.997	0.0085

20	74.000	74.010	74.013	74.020	74.003	0.0080
21	74.982	74.001	74.015	74.005	73.996	0.0122
22	74.004	73.999	73.990	74.006	74.009	0.0074
23	74.010	73.989	73.990	74.009	74.014	0.0119
24	74.015	74.008	73.993	74.000	74.010	0.0087
25	73.982	73.984	73.995	74.017	74.013	0.0162

n = 5 and $\bar{S}$ = 0.0094

4.1a. Three Sigma Control Limits for Standard Deviation Chart

The 3σ control limits suggested by Shewhart (1931) are $\bar{S} \pm \left\{ \left(3\sqrt{1-c_4^{\,2}}\right)\left(\bar{S}/c_4\right) \right\}$

For n = 4, $c_4 = 0.9400$ (Quality control factor, W.A.Shewhart)

$$\mathrm{UCL}_{3\sigma} = \bar{S} + \left\{ \left(3\sqrt{1-c_4^{\,2}}\right)\left(\bar{S}/c_4\right) \right\} = 0.0094 + \left\{ \left(3\sqrt{1-0.9400^2}\right)(0.0094/0.9400) \right\}$$
$$= 0.0094 + 0.0102 = 0.0196$$
$$\text{Central Line } \mathrm{CL}_{3\sigma} = \bar{S} = 0.0094$$
$$\mathrm{LCL}_{3\sigma} = \bar{S} - \left\{ \left(3\sqrt{1-c_4^{\,2}}\right)\left(\bar{S}/c_4\right) \right\} = 0.0094 - \left\{ \left(3\sqrt{1-0.9400^2}\right)(0.0094/0.9400) \right\}$$
$$= 0.0094 - 0.0102 = 0$$

From the resulting Figure 1 that the process is in control, since all the samples lie inside the control limits.

4.1b. Control Limits Based on Six Sigma Initiatives for Standard Deviation Chart

For a given TL = 0.0133 (USL-LSL =0.0162-0.0029) & C_p = 1.5, it is found from the Table 1 that the value of $\sigma_{6\sigma}$ is 0.0015. The control limits based on six sigma initiatives for Standard deviation chart for a specified TL and $O_{6\sigma}$ are $\bar{S} \pm \left\{ \left(4.831 \times \sqrt{1-c_4^{\,2}}\right) \times \sigma_{6\sigma} \right\}$ with

$$UCL_{6\sigma} = \bar{S} + \left\{ \left(O_{6\sigma} \times \sqrt{1-c_4^{\,2}}\right) \right\} \times \sigma_{6\sigma} = 0.0094 + \left\{ \left(4.831 \times \sqrt{1-0.9400^2}\right) \times 0.0015 \right\} = 0.0119$$
$$Center\ line\ CL_{6\sigma} = \bar{S} = 0.0094$$
$$LCL_{6\sigma} = \bar{S} - \left\{ \left(O_{6\sigma} \times \sqrt{1-c_4^{\,2}}\right) \right\} \times \sigma_{6\sigma} = 0.0094 - \left\{ \left(4.831 \times \sqrt{1-0.9400^2}\right) \times 0.0015 \right\} = 0.0069$$

From the resulting Figure 1 that the sample numbers 1, 3, 5, 8, 14, 21 and 25 goes above the upper control limit and the sample numbers 7, 9, 10, 11 and 12 goes below the lower control limit. Therefore the process does not exhibit statistical control.

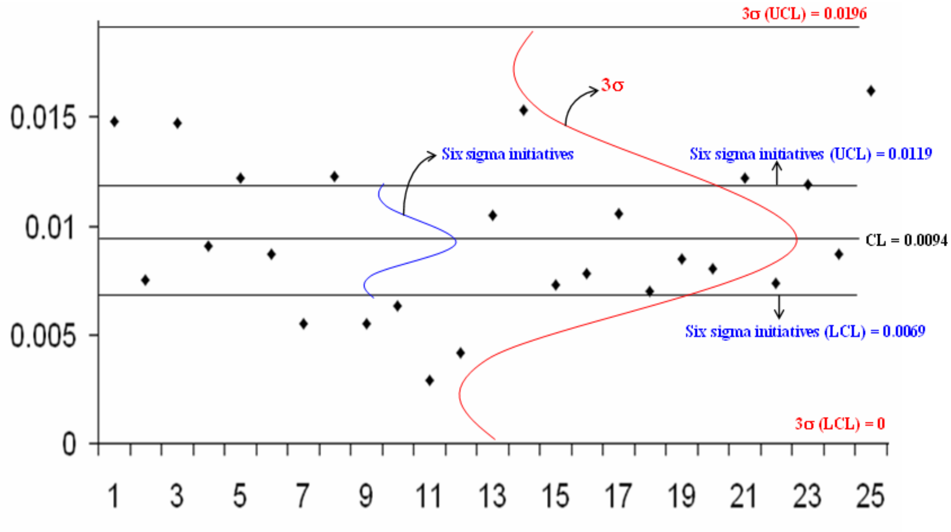

Figure 1: *Comparison of the process: 3σ limits and control limits using Six Sigma initiatives*

5. Conclusion

In this paper, a procedure is given to construct a control chart based on six sigma initiatives for Standard deviation with an example. It is found that the process was not in control even when Six Sigma initiatives are adopted. It is very clear from the comparison that when the process is centered with reduced variation many points fall outside the control limits than the 3 sigma control limits, which indicate that the process is not in the level it was expected. So a correction in the process is very much required to reduce the variations. In future, all the companies are adopting this technique instead of the existing Shewhart chart in their organization.

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Research Article **Open Access**

Construction of Super-Saturated Designs

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Abstract A design which is having more number of factors than the number of design points is called a Super-saturated Design. In this paper, an attempt is made to propose two new series of constructions of super-saturated designs using mutual orthogonal Latin squares and balanced incomplete block designs and the methods are illustrated with suitable examples.
Keywords *Balanced Incomplete Block Design; Mutual Orthogonal Latin Squares; Super-Saturated Design*

1. Introduction

If the number of factors or factor combinations is more, and only few measured are interest to study, then eliminate the insignificant factor combinations which are not affecting much the response that is the loss of information due to the elimination of factors should be as minimum as possible. High-dimensional datasets makes many mathematical challenges in fitting and give rise to new theoretical developments, which reduces the time, cost, effort and complexity by reducing its dimension. To minimize the number of design points, identify the active factors for efficient utilization of resources. The knowledge of each and every main effect may not useful since insignificant factors are not usually of interest. Reduce the number of design points or chose the design with minimum design points and optimum. These designs reduce the experimental cost and time significantly due to their run size.

Scatterthwaite (1959) initially constructed a new class of balanced designs randomly with a property that number of design points is equal to the number of factors and Booth and Cox (1962) proposed a systematic method of construction of designs in which the number of factors exceeds the number of design points. Let ‘p’ be the number of factors and ‘n’ be the number of design points in a design. Then the saturated and super saturated designs can be defined as

Definition 1.1: A design X is said to be ‘saturated’ if the number of design points ‘n’ is equal to the number of factors ‘p’ plus one i.e. $n=p+1$.
Definition 1.2: A design X is said to be a super-saturated design, if the number of factors “p” is more than the number of design points ‘n’ i.e. $p > n$.

Designs satisfying the orthogonality are more preferred due to their optimality. If it is not possible to conduct the experiment with orthogonal design, search for designs that are near orthogonal. For such designs, the lack of orthogonality can be measured based on the covariance's.

Definition 1.3: A design X is said to be $E(s^2)$-optimal super-saturated if a super-saturated design possessing the property that the mean of s_{ij}^2 of all pairs (i, j) for (i≠ j) is minimum.

2. Literature Review on Super-Saturated Designs

After Booth and Cox (1962), several authors like Lin (1993), Nguyen (1996), Cheng (1997), Li and Wu (1997) Tang and Wu (1997), Deng, Lin and Wang (1999), Fang, Lin and Ma (2000), Liu and Zhang (2000), Lu and Sun (2001), Butler, Mead, Eskridge and Gilmour (2001), and Yamada and Lin (2002), Li and Lin (2003), Liu and Dean (2004), Fang, Gennian and Liu (2004), Aggarwal and Gupta (2004), Xu and Wu (2005), Koukouvinov, Mantos and Mylona (2007), Jones, Lin and Nachtsheim (2008), Nguyen and Cheng (2008), Sun, Lin and Liu (2011), Liu and Liu (2012), Gupta and Morales (2012), Chatterjee, K., Koukouvinos, C, Mantas, P and Skountzou, A (2012), Hung, C., Lin, D.K.J., and Liu M.Q. (2012), Mbegbu, J.I., and Todo, C.O. (2012), Liu, Y. and Liu, M.Q. (2013), Miller, A., and Tang, B, (2013), Ameen Saheb and Bhatracharyulu (2013, 2014) etc made attempts on the construction of super-saturated designs with their $E(s^2)$ optimality.

Lin (1993), Li and Wu (1997) motivated towards the construction of super-saturated designs through column wise, pair wise exchanges. They differ from the k-exchange algorithms in two aspects, one is, they exchange columns instead of rows of the design matrix and another one is, they employ a pair wise adjustment in the search for a better column. Deng, Lin and Wang (1999) studied the properties of super-saturated designs and proposed a criterion based on the projection property called resolution rank.

Fang, Lin, and Ma (2000) proposed a construction procedure by embedding a saturated orthogonal design into a uniform design of the same row size. They adopt the collapsing method from Addelman. The basic idea of the construction method is, to collapse a multi-level factor into several low-level factors, where they collapse U-type uniform designs. They proposed five criteria for comparing multi-level super-saturated designs.

Liu and Zhang (2000) proposed a general algorithm for the construction of $E(s^2)$ optimal super-saturated designs from cyclic BIBD. The general formula for the lower bound of a super-saturated design with 'm' factors with 'n' design points is $[n^2(m-n+1)] / [(n-1)(m-1)]$. Lu and Sun (2001) proposed two criteria denoted by $E(s^2)$ and $\max\{(s^2)\}$ in the construction of multi-level super-saturated designs. Eskridge, Gilmour, Mead, Butler and Travnicek (2001) and Liu and Dean (2002) also considered cyclic generation of $E\ (s^2)$-optimal and nearly optimal super-saturated designs.

Fang, Gennian and Liu (2002) proposed a discrete discrepancy as a measure of uniformity for super-saturated designs and a lower bound of this discrepancy is obtained as a benchmark of design uniformity and also proposed construction procedure for uniform super-saturated by using resolvable BIBD along with their properties. Yamada and Lin (2002) suggested a construction method for mixed-level super-saturated designs consisting of two-level and three-level columns. The chi-square statistics is used for a measure of dependency of the design columns. The dependency properties for the newly constructed designs are derived and discussed. Li and Lin (2002) proposed variable selection procedure to screen active effects in the super-saturated designs via non-convex penalized least squares approach and empirical comparison with Bayesian variable selection approaches is made.

Liu and Dean (2004) proposed a class of super-saturated designs called k- Circulant super-saturated designs which can be obtained from cyclic development of a generator. This method is a generalization of Plackett-Burman, who introduced the use of cyclic generators for constructing orthogonal saturated designs.

Fang Kaitai, Gennian G.E., and Liu Minqian (2004) proposed a combinatorial approach called the packing method. They studied the connection between orthogonal arrays and resolvable packing designs for constructing optimal super-saturated designs and properties of the resulting designs are also proposed. Aggarwal and Gupta (2004) proposed construction method for multi-level super-saturated designs based on Galois field theory. Xu and Wu (2005) proposed construction methods for multi-level super-saturated designs inspired by Addelman- Kempthorne of orthogonal arrays and also proposed new lower bound for multi-level super-saturated designs.

Koukouvinov, Mantas and Mylona (2007) proposed mixed-level super-saturated designs by using supplementary difference sets with respect to the $E(f_{NOD})$ criterion. Nguyen and Cheng (2008) suggested the construction procedure for super-saturated designs from BIBD and also from regular graph designs when BIBD do not exist. Jones, Lin and Nachtsheim (2008) proposed a new class of super-saturated designs by using Bayesian D-optimality for arbitrary sample sizes and for any number of blocks of any size and also incorporate categorical factors with more than two levels.

Sun, Lin and Liu (2011) proposed equivalent conditions for two columns to be fully aliased and consequently proposed methods for constructing $E(f_{NOD})$ and chi-square optimal mixed level super-saturated designs without fully aliased columns via equidistant designs and difference matrices.

Gupta, Hisano and Morales (2011) proposed a systematic method of construction for optimal k-circulant multi-level supersaturated designs and also constructed two-level designs using resolvable BIBD. Liu, Y., and Liu, M.Q., (2011) proposed a new method for constructing mixed-level designs with relatively large number of levels avoiding the blind search and numerous calculations by computers. The goodness of the resulting supersaturated design is judged by the χ^2 and J_2 criteria's.

Mandal B.N., Gupta, V.K., and Prasas, R., (2011) proposed an algorithm to construct efficient balanced multi-level K-circulant supersaturated designs with 'm' factors and 'n' runs and they also constructed 60 factor and 10 levels multi-level supersaturated designs.

Liu and Liu (2012) generalized a method proposed by Liu and Lin to the mixed level case and also they proposed two new practical methods for constructing optimal mixed level supersaturated designs.

Gupta (2012) extended the work from two level to s-level balanced supersaturated designs. Gupta and Morales (2012) proposed tabu search method for constructing $E(s^2)$-optimal and minimax – optimal k-circulant supersaturated designs. Chatterjee, K., Koukouvinos C., Mantas, P., and Skountzou, A. (2012) proposed $E(f_{NOD})$-optimal multi-level supersaturated designs with a large number of columns based on the new supplementary difference sets method.

Hung, C., Lin, D.K.J., and Liu M.Q. (2012) proposed a new criterion for supersaturated designs with quantitative factors. Mbegbu, J.I., and Todo, C.O. (2012) proposed $E(s^2)$-optimal supersaturated designs with an experimental run size n=20 and number of factors m=57(multiple of 19). This construction is based on BIBD using a theorem proposed by Bulutoglu and Cheng.

Liu, Y., and Liu, M.Q. (2013) proposed complementary design method. The basic principle of this method is that for any existing $E(f_{NOD})$ optimal supersaturated whose $E(f_{NOD})$ value reaches its lower bound, its complementary design in the corresponding maximal balanced design is also $E(f_{NOD})$ optimal. This method applies to both symmetrical and asymmetrical cases.

Miller, A., and Tang, B. (2013) proposed supersaturated designs using minimal dependent sets (MDS) of columns in the design matrix. Ameen Saheb and Bhatracharyulu (2013, 2014) proposed two level supersaturated designs by using cyclic resolvable designs and also they proposed two new methods for constructing supersaturated designs using row-column and cyclic resolvable designs.

In this paper, an attempt is made propose to construct supersaturated designs using Balanced 'n' array Block Designs and Balanced Incomplete Block Designs.

3. Construction of Two Level Super-Saturated Designs

In this section, an attempt is made propose to construct two level supersaturated designs using Balanced n-array Block Design's (BnBD) and Balanced Incomplete Block Designs with suitable illustrations is presented. The $E(s^2)$-optimal values for the designs are also evaluated.

Method 3.1: Consider a complete set of (n-1) mutually orthogonal Latin squares (MOLS) of order n, where 'n' is prime or prime power. Arrange the (n-1) MOLS together to form an array of order n(n-1)x n consisting of elements p_0, p_1, … , p_{n-1} in each row. Replace 'n_1' of the p_i's by +1's, 'n_2' of the p_i 's by -1's such that $n_1 + n_2 = n$. By considering each row of n(n-1) corresponds to a factor and each column as a design point the resulting design is a super-saturated design.

Note: When 'n' is even, complete set of mutually orthogonal Latin squares can be considered and when 'n' is odd either all (n-1) MOLS or a half of complete set, i.e. (n-1)/2 MOLS such that ${}^nC_2$ pairs of the 'n' elements occur exactly once in any two columned sub matrix of the array of order n(n-1)/2 x n can be considered.

Example 3.1: Consider the balanced 5 array design in 10 blocks replace p_0 and p_1 by '-1' and the other p_i's '+1'. The resulting supersaturated design with 5 design points and 10 factors is given below.

$$X=\begin{bmatrix} -1 & -1 & +1 & +1 & +1 & -1 & -1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 & -1 & +1 & +1 & -1 & -1 & +1 \\ +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 \\ +1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 \end{bmatrix}$$

The expected value of s^2 of the design is 3.6.

Method 3.2: Consider a Balanced incomplete Block Design with parameters v, b, r, k and λ whose incidence matrix is N^*. Obtain the matrix N_1 0's with -1 in N^* and obtain the matrix N_2 by replacing '1' with '-1' and 0's with 1's. A super saturated design X can be constructed with 'v' design points in '2b' factors as $X = [N_1 \quad N_2]'$ where N_2 is replacing '1' by '-1' and '-1' by '1' of N_1.

Remark: When v=2k, X′X is singular then design matrix X has to be modified suitably given by $X^*=[X \quad J \quad -J]'$ where 'J' be a vector of unities.

Example 3.2: Consider a BIBD with parameters v=7=b, r=3=k, λ=1 whose incidence matrix is N^*. By delivering the matrices N_1 and N_2 by replacing the 0 by -1 and 0 by 1 and 1 by -1 respectively. The resulting supersaturated design is

$$X=\begin{bmatrix} +1 & +1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & +1 \\ -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 \\ -1 & -1 & +1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & +1 & +1 & -1 \\ -1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & +1 & -1 \\ -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 \end{bmatrix}$$

The expected value of s^2 of the design is 4.65.

4. Concluding Remarks

Using the proposed methods, it is possible to estimate maximum number of main effects when compared to any other super-saturated designs. It found that the proposed methods of supersaturated designs are more efficient than the existed Booth and Cox designs.

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Methodology Article **Open Access**

The Multivariate Central Limit Theorem and its Relationship with Univariate Statistics

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Abstract In the present article the multivariate central limit theorem is revisited. Rather than simply reviewing existing methodology our approach mostly aims at giving particular emphasis on some univariate techniques that support the proof of this theorem. On those grounds all necessary mathematical arguments are duly provided.
Keywords *Characteristic Function; Convergence in Distribution; Normal Distribution*

1. Introduction

The well-known central limit theorem for independent and identically distributed vector random variables is a very important result since it allows for an approximate normal distribution for the mean (and, equivalently, for the sum) of a sequence of a very large number of variables which satisfy the aforementioned properties; thus it has been extensively mentioned and used in existing literature. The idea of the present article emanates from Theorem 29.4 of [2, p. 383], and a subsequent comment, namely that "certain limit theorems can be reduced in a routine way to the one-dimensional case". Based on that comment our intention is therefore to revisit the proof of this theorem in view of providing a detailed description of the way statistical theory that is used in the one-dimensional case (mentioned in the text as the univariate case) applies in a straightforward manner in order to deal with a multivariate problem.

2. The Multivariate Central Limit Theorem

The multivariate central limit theorem we will focus on is called Theorem 1 and is an early result which is very famous and well documented. For instance, it is found as Theorem 3.4.3 in [1, p. 81]. We first state this result.

Theorem 1

Let $X_1, X_2, \ldots$ *be a sequence of p-dimensional independent and identically distributed random vectors with finite mean vector* $E(X_j) = \mu$ *and finite variance-covariance matrix* $E(X_j - \mu)(X_j - \mu)' = \Sigma$. *Then the asymptotic distribution of* $\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(X_j - \mu)$ *converges towards a normal* $\mathrm{N}(\underset{\sim}{0}, \Sigma)$ *as* $n \to \infty$.

Remark that in the univariate case, i.e. the case where *p*=1, Theorem 1 is just the univariate central limit theorem also called the Lindeberg-Lévy theorem (see for example [3, p. 215]). The proof of the multivariate central limit theorem, which we provide in the sequel, will be essentially based on techniques appearing in the proof of Lindeberg-Lévy's theorem.

Proof of Theorem 1

Like in Theorem 3.4.3 of [1, p. 81], we use $\phi(t,s) = [E(e^{is\frac{1}{\sqrt{n}}\sum_{j=1}^{n}[tX_j - E(tXj)]})]$ as characteristic function of $\frac{1}{\sqrt{n}}\sum_{j=1}^{n}[tX_j - E(tX_j)]$, for real *s*, fixed $t = (t_1, \ldots, t_p)$, and $i^2 = -1$. Also letting $X = \sum_{j=1}^{n} tX_j$, with $E(tX_j) = m$ and $Var(tX_j) = \sigma^2$ we obtain $E(X) = m_X = E(\sum_{j=1}^{n} tX_j) = nm$, and $Var(X) = \sigma_X^2 = Var(\sum_{j=1}^{n} tX_j) = n\sigma^2$ (or in other words $\sigma_X = \sigma\sqrt{n}$), since the X_j's are independently distributed. We now adopt the same rationale as that used by [3, p. 215] for the proof of the Lindeberg-Lévy theorem. Let $\hat{\phi}(u)$ be the characteristic function of $tX_j - m$ and $\tilde{\phi}(u)$ the characteristic function of $X - m_X$. We have

$$\tilde{\phi}(u) = E(e^{iu(X-m)}) = E(e^{iu[(tX_1 - m)+\ldots+(tX_n - m)]}) = e^{-iunm}E(e^{iutX_1}\ldots e^{iutX_n}),$$

and since the X_j's are independently distributed we obtain $\tilde{\phi}(u) = e^{-ium}E(e^{iutX_1})\ldots e^{-ium}E(e^{iutX_n}) = \hat{\phi}(u)\ldots\hat{\phi}(u) = [\hat{\phi}(u)]^n$ (since $\hat{\phi}(u) = e^{-ium}E(e^{iutX_j})$, for any $j = 1, \ldots n$). Then, if $\phi(u)$ is the characteristic function of $(X - m_X)/\sigma_X$, we obtain $\phi(u) = E(e^{iu(\frac{X - m_X}{\sigma_x})}) = E(e^{i\frac{u}{\sigma_X}(X - m_X)}) = \tilde{\phi}(\frac{u}{\sigma_X})$, with $\tilde{\phi}(\frac{u}{\sigma_X}) = [\hat{\phi}(\frac{u}{\sigma_X})]^n$ (since $\tilde{\phi}(u) = [\hat{\phi}(u)]^n$).

Thus $\phi(u) = [\hat{\phi}(\frac{u}{\sigma\sqrt{n}})]^n$.

On the other hand $\hat{\phi}(u) = \int_{-\infty}^{\infty} e^{iuy} f(y)dy$, where $Y_j = tX_j - E(tX_j)$, and by second order Taylor's expansion around 0 we obtain

$$\hat{\phi}(u) = \hat{\phi}(0) + (u-0)\hat{\phi}'(0) + [(u-0)^2/2]\hat{\phi}''(0) + o(u-0)^2 = \hat{\phi}(0) + u\hat{\phi}'(0) + (u^2/2)\hat{\phi}''(0) + o(u^2)$$

where $\hat{\phi}'(u) = i\int_{-\infty}^{\infty} ye^{iuy} f(y)dy$ and $\hat{\phi}''(u) = i^2\int_{-\infty}^{\infty} y^2e^{iuy} f(y)dy = -\int_{-\infty}^{\infty} y^2e^{iuy} f(y)dy$.

Hence we have $\hat{\phi}(0) = \int_{-\infty}^{\infty} f(y)dy = 1$, $\hat{\phi}'(0) = i\int_{-\infty}^{\infty} yf(y)dy = iE(Y_j) = 0$, and

$\hat{\phi}''(0) = -\int_{-\infty}^{\infty} y^2 f(y)dy = -E(Y_j^2)$, and so we obtain $\hat{\phi}(u) = 1-[(u)^2/2]E(Y_j^2)+o(u^2)$.

Finally, since $E(Y_j^2) = Var(tX_j) = \sigma^2$, we obtain $\hat{\phi}(u) = 1-(\sigma^2 u^2/2)+o(u^2)$.

Hence $\phi(u) = [\hat{\phi}(\frac{u}{\sigma\sqrt{n}})]^n = [1-(\sigma^2 u^2/2n\sigma^2)+\eta(n,u)/n]^n = (1-u^2/2n+\eta(n,u)/n)^n$

where, for every fixed u, $\eta(n,u) \to 0$ as $n \to \infty$.

We now let $x = -u^2/2n+\eta(n,u)/n$; then $\lim_{n\to\infty}(x) = 0$ and

$$\lim_{n\to\infty}(1-u^2/2n+\eta(n,u)/n)^n = \lim_{x\to 0}(1+x)^{-u^2/2x}\lim_{x\to 0}(1+x)^{\eta(n,u)/x}$$

$$= \{\lim_{x\to 0}[(1+x)^{1/x}]^{-u^2/2}\}\{\lim_{x\to 0}[(1+x)^{1/x}]^{\eta(n,u)}\}.$$

Since a well known result is that $\lim_{x\to 0}[(1+x)^{1/x}] = e$, and on the other hand $n\to\infty$ as $x\to 0$ imply that $\eta(n,u) \to 0$ as $x\to 0$, for fixed u, we obtain $\lim_{n\to\infty}(1-u^2/2n+\eta(n,u)/n)^n = e^{-u^2/2}$, and thus $\lim_{n\to\infty}\phi(u) = e^{-u^2/2}$, for any u, $e^{-u^2/2}$ being continuous at $u=0$, with $\lim_{n\to\infty}\phi(0) = 1$, and so we deduce that ϕ is itself a characteristic function (in other words we are not in presence of a pathological situation such as that described in exercise 5.12.35 and solution of [5, p. 266]). It results from these arguments that $\phi(u)$ is indeed the characteristic function of $(X-m_X)/\sigma_X$. On the other hand $e^{-u^2/2}$ is the characteristic function of a standard normal random variable and hence, by the continuity theorem for characteristic functions concerning the univariate case (see for instance [4, p. 190]), $(X-m_X)/\sigma_X$ converges in distribution towards a N(0,1) random variable as $n\to\infty$. This is equivalent to saying that $(X-nm)/\sqrt{n}$ converges in distribution towards N(0, σ^2), with $\sigma^2 = Var(tX_j) = t\Sigma t'$ (where prime denotes transpose), and thus $\frac{1}{\sqrt{n}}\sum_{j=1}^{n}[tX_j - E(tX_j)]$ converges in distribution towards a N(0, $t\Sigma t'$) random variable (see also proof of Theorem 3.4.3 of [1, p. 81]). The characteristic function of the latter variable is as given by [4, p. 187], Example (5), and reduces to $e^{\frac{-1}{2}s^2 t\Sigma t'}$ for our case since the mean is 0. Thus we have that for each $(t_1,...,t_p)$

$$\lim_{n\to\infty}\phi(t,s) = \lim_{n\to\infty}E(e^{is\frac{1}{\sqrt{n}}\sum_{j=1}^{n}[tX_j-E(tXj)]}) = e^{\frac{-1}{2}s^2 t\Sigma t'}.$$

Like in [2, p. 383] or in [1, p. 81], we take $s=1$ in order to obtain

$$\lim_{n\to\infty}\phi(t,1) = \lim_{n\to\infty}E(e^{i\frac{1}{\sqrt{n}}\sum_{j=1}^{n}[tX_j-E(tXj)]}) = e^{\frac{-1}{2}t\Sigma t'}.$$

Note that $\phi(t,1) = E(e^{i\frac{1}{\sqrt{n}}\sum_{j=1}^{n}[tX_j - E(tXj)]}) = E(e^{it\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(X_j-\mu)})$ is the characteristic function of $\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(X_j - \mu)$. On the other hand $\lim_{n\to\infty}\phi(t,1) = e^{\frac{-1}{2}t\Sigma t'}$ is continuous at $t = (t_1,...,t_p) = (0,...,0) = \underset{\sim}{0}$, with $\lim_{n\to\infty}\phi(\underset{\sim}{0},1) = 1$, and so by means of the continuity theorem for characteristic functions that concern the multivariate case (see Theorem 2.6.4 of [1, pp. 48-49]) we have that $\lim_{n\to\infty}\phi(t,1)$ is identical with the characteristic function $e^{\frac{-1}{2}t\Sigma t'}$ of a normally distributed vector random variable N($\underset{\sim}{0}$,Σ) (see [5, p. 187]), Example (6), or Theorem 2.6.1 of [1, p. 45], for documentation on the characteristic function $e^{\frac{-1}{2}t\Sigma t'}$). We thus conclude that $\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(X_j - \mu)$ converges in distribution towards a normally distributed N($\underset{\sim}{0}$,Σ) vector random variable, and hence Theorem 1 is proved.

3. Conclusion

The above arguments underline the obvious implication (and thus the crucial role) of univariate statistical techniques for proving multivariate results. These techniques arise from the use of the random variable $\frac{1}{\sqrt{n}}\sum_{j=1}^{n}[tX_j - E(tX_j)]$ and its characteristic function $\phi(t,s)$ in the proof of Theorem 1. Remark that they are also involved in the proof of Lindeberg-Lévy's theorem. Finally it is noteworthy to stress that the idea of using the variable $\frac{1}{\sqrt{n}}\sum_{j=1}^{n}[tX_j - E(tX_j)]$ and its characteristic function in the preceding section occurs naturally in view of Theorem 29.4 of [2, p. 383].

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Research Article

On ω-Convergence of *p*-Stacks

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Abstract We introduce the notion of ω-convergence of *p*-stacks and by using that notion we characterize the ω-interior, ω-closure, separation axioms and ω-irresoluteness on a topological space.
Keywords *Topological spaces; ω-open sets; ω-closed spaces*

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Sundaram and Sheik John [5] introduced a new class of generalized open sets called ω-open sets into the field of topology. In this paper, we have introduced and study the notion of ω-convergence of p-stacks and by using that notion we characterize the ω-interior, ω-closure, separation axioms and ω-irresoluteness on a topological space. Also we have introduced a new notion of p-ω-compactness and investigate its properties in terms of ω-convergence of p-stacks.

2. Preliminaries

Throughout this paper, spaces always means topological spaces on which no separation axioms are assumed unless otherwise mentioned and $f : (X,\tau) \to (Y,\sigma)$ (or simply $f : X \to Y$) denotes a function f of a space (X,τ) into a space (Y,σ). Let A be a subset of a space X. The closure and the interior of A are denoted by Cl(A) and Int(A), respectively. A subset A of a space (X,τ) is called semi open [1] if $A \subset Cl(Int(A))$. A subset A of a space X is called ω-closed [5] if $Cl(A) \subset U$ whenever $A \subset U$ and U is semi-open in X. The complement of an ω-closed set is called an ω-open set. The family of all ω-open subsets of (X,τ) is denoted by $\omega(\tau)$. We set $\omega(X,x) = \{V \in \omega(\tau) | x \in V\}$ for $x \in X$. The union (resp. intersection) of all ω-open (resp. ω-closed) sets, each contained in (resp. containing) a set A in a space X is called the ω-interior (resp. ω-closure) of A and is denoted by ω Int(A) (resp. ω Cl(A)) [4]. A subset M(x) of a topological space X is called a ω-neighbourhood of a point $x \in X$ if there exists a ω-open set S such that $x \in S \subset M(x)$. Given a set X, a collection C of subsets of X is called a stack if $A \in C$ whenever $B \in C$ and $B \subset A$. A stack H on a set X is called a p-stack if it satisfies the following condition: (P) $A,B \in H \Rightarrow A \cap B \neq \emptyset$. Condition (P) is called the pairwise intersection property (P.I.P). A collection B of subsets of X with the P.I.P is called a p-stack base. For any collection B, we denote by $\langle B\rangle = \{A \subset X$: there exists $B \in B$ such that $B \subset A\}$ the stack generated by B, and if {B} is a p-stack base, then $\langle\{B\}\rangle$ is a p-stack. We will denote simply $\langle\{B\}\rangle = \langle B\rangle$. In case $x \in X$ and $B = \{x\}$, $\langle x\rangle$ is usually denoted by ˙x. Let pS(X) denote the collection of all p-stacks on X, partially ordered by inclusion. The maximal elements in pS(X) are called ultrapstacks is contained in an ultrapstack. For a function f : X → Y and H ∈ pS(X), the image stack f(H) in pS(Y) has p-stack base {f(H) : H ∈ H}. Likewise, if G ∈ pS(Y), f−1(G) denotes the p-stack on X generated by {f−1(G) : G ∈ G}.

Definition 2.1. Let (X,τ) be a topological space. A class {Gi} of ω-open subsets of X is said to be ω-open cover of X if each point in X belongs to atleast one Gi that is∪ i Gi = X.

Definition 2.2. A subset K of a nonempty set X is said to be ω-compact [4] relative to (X,τ) if every cover of K by ω-open sets of X has a finite subcover. We say that (X,τ) is ω-compact if X is ω-compact.

Definition 2.3. A topological space (X,τ) is said to be:

(i) ω-T1 [3] if for each pair of distinct points x and y of X, there exist ω-open sets U and V containing x and y, respectively such that y / ∈ U and x / ∈ V .
(ii) ω-T2 [3] if for each pair of distinct points x and y of X, there exist ω-open sets U and V such that x ∈ U, y ∈ V and U ∩V = ∅.
(iii) ω-regular [4] if for any closed set F ⊂ X and any point x ∈ X\F, there exist disjoint ω-open sets U and V such that x ∈ U and F ⊂ V .

Lemma 2.4. [2] For H ∈ pS(X), the following are equivalent:

(i) H is an ultrapstack.
(ii) If A∩H /= ∅ for all H ∈ H, then A ∈ H;
(iii) B ∈ H implies X\B ∈ H.

Theorem 2.5. [2] Let f : (X,τ) → (Y,σ) be a function and H ∈ pS(X).

(i) If H is a filter, so is f(H);
(ii) If H is an ultrafilter, so is f(H);
(iii) If H is an ultrapstack, so is f(H).

3. ω-convergence of p-stacks

Definition 3.1. Let X be a topological space, x ∈ X and let B(x) = {V ⊂ X: V is a ω-neighbourhood of x}. Then we call the family B(x) the ω-neighbourhood stack at x.

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Theorem 3.3. Let (X,τ) be a topological space. Then we have the following

(i) x ω-converges to x for all x ∈ X.
(ii) If F ω-converges to x and F ⊂ G for F,G ∈ pS(X), then G ω-converges to x.
(iii) If both F and G are p-stacks ω-converging to x, then F ∩ G ω-converges to x.
(iv) If p-stacks Fi ω-converge to x for all i ∈ J, then ∩Fi ω-converges to x.

Proof. Follows from the definitions.

Theorem 3.4. Let (X,τ) be a topological space and $A \subset X$. Then the following are equivalent:

(i) $x \in \omega\, Cl(A)$;
(ii) There is $F \in pS(X)$ such that $A \in F$ and F ω-converges to x;
(iii) For all $V \in B(x)$, $A \cap V \neq \emptyset$.

Proof. (i) ⇒ (ii): Let x be an element in $bCl(A)$, then $U(x) \cap A = \emptyset$ for each ω-open $U(x)$ of x. Let $F = B(x) \cup \langle A \rangle$. Then the p-stack F ω-converges to x and $A \in F$. (ii) ⇒ (iii): Let F be a p-stack and $A \in F$ and p-stack F ω-converge to x. Then $B(x) \subset F$. Thus since $B(x)$ is a p-stack, we get $U \cap A \neq \emptyset$ for all $U \in B(x)$. (iii) ⇒ (i): It is obvious.

Theorem 3.5. Let (X,τ) be a topological space and $A \subset X$. Then the following are equivalent:

(i) $x \in \omega\, Int(A)$;
(ii) For every p-stack F ω-converging to x, $A \in F$;
(iii) $A \in B(x)$.

Proof. (i) ⇒ (ii): Let x be an element in $bInt(A)$ and let F be a p-stack ω-converging to x. Since $x \in bInt(A)$, there is a ω-open subset U such that $x \in U \subset A$, so $A \in B(x)$. Thus by the definition of ω-convergence of p-stack, we can say $A \in F$. (ii) ⇒ (iii): The ω-neighborhood stack $B(x)$ is always ω-converges to x. Thus by (ii), $A \in B(x)$. (iii) ⇒ (i): It is obvious.

Now by using ω-convergence of p-stacks, we characterize the properties of ω-T1, ω-T2 and ω-regular induced by ω-open subsets on a topological space.

Theorem 3.6. Let (X,τ) be a topological space. Then the following statements are equivalent:

(i) (X,τ) is ω-T1;
(ii) $\cap B(x) = \{x\}$ for $x \in X$;
(iii) If $\dot{x}$ ω-converges to y, then $x = y$.

Proof. (i) ⇒ (ii): Let y be an element in $\cap B(x)$, then $y \in U$ for each ω-open neighborhood U of x. Since X is ω-T1, we get $y = x$. (ii) ⇒ (iii): Let $\dot{x}$ ω-converge to y. Since $B(y) \subset \dot{x}$, x is an element in $\cap B(y)$. Thus $x = y$. (iii) ⇒ (i): Suppose that X is not ω-T1, then there are distinctx and y such that every ω-open neighborhood of x contains y. Thus $B(x) \subset \dot{y}$ and $\dot{y}$ ω-converges to x. This contradicts the hypothesis.

Theorem 3.7. Let (X,τ) be a topological space. Then the following statements are equivalent:

(i) (X,τ) is ω-T2;
(ii) Every ω-convergent p-stack F on X ω-converges to exactly one point;
(iii) Every ω-convergent ultrapstack F on X ω-converges to exactly one point.

Proof. (i) ⇒ (ii): Suppose that X is ω-T2 and a p-stack F ω-converges to x. For any $y \neq x$, there are disjoint ω-open sets $U(x)$ and $U(y)$ containing x and y, respectively. Since $B(x) \subset F$ and F is a p-stack, both $U(x)$ and $X \setminus U(y)$ are elements of F. Thus F is not finer than (y), so F doesn't ω-converge to y. (ii) ⇒ (iii): It is obvious. (iii) ⇒ (i): Suppose that X is not ω-T2. Then there must exist x, y such that $U(x) \cap U(y) \neq \emptyset$ for every ω-open sets $U(x)$ and $U(y)$ of x and y, respectively. Let F be a ultrapstak finer than a p-stack $B(x) \subset B(y)$. Then F is finer than $B(x)$ and (y), so the ultrapstack F ω-converges to both x and y. This contradicts (ii). If (X,τ) is a topological space and $F \in pS(X)$, then $B = \{\omega\, Cl(F) : F \in F\}$ is a p-stack base on X, and the ω-closure p-stack generated by B is denoted by $\omega\, Cl(F)$.

Theorem 3.8. Let (X,τ) be a topological space. Then the following statements are equivalent:

(i) (X,τ) is ω-regular;
(ii) For every x ∈ X, B(x) = ω Cl(B(x));
(iii) If a p-stack F ω-converges to x, then the ω-closure p-stack ω Cl(F) ω-converges to x.

Proof. (i) ⇒ (ii): Let F be an element in B(x). There exists a ω-open neighborhood U(x) such that U(x) ⊂ F. Since X is ω-regular, there is a ω-open neighborhood W(x) of x such that W(x) ⊂ ω Cl(W(x)) ⊂ U(x) ⊂ F. Since ω Cl(W(x)) ∈ ω Cl(B(x)) and ω Cl(B(x)) is a pstack, F ∈ ω Cl(B(x)). (ii) ⇒ (iii): Let a p-stack F ω-converge to x. Then B(x) ⊂ F, and so ω Cl(B(x)) ⊂ ω Cl(F). By (ii), we get that ω Cl(F) ω-converges to x. (iii) ⇒ (i): Let U be a ω-open set containing x ∈ X. Since B(x) ω-converges to x, by (iii) ω Cl(B(x)) ω-converges to x, and so U ∈ ω Cl(B(x)). Then by the definition of the ω-closure of p-stacks, we can get a ω-open neighborhood V of x such that V ⊂ ω Cl(V) ⊂ U.

Definition 3.9. A function f: (X,τ) → (Y,σ) is said to be ω-irresolute [4] if f−1(V) is ω-closed (resp. ω-open) in X for every ω-closed (resp. ω-open) subset V of Y.

Theorem 3.10. Let X and Y be topological spaces. Then a function f : (X,τ) → (Y,σ) is ω-irresolute if and only if for each x in X and each ω-neighborhood U of f(x), there is a ω-neighborhood V of x such that f(V) ⊂ U. Now we get another characterization of the ω-irresolute function on a topological space using the notion of p-stacks.

Theorem 3.11. For a function f : (X,τ) → (Y,σ), the following statements are equivalent: (i) f is ω-irresolute; (ii) B(f(x)) ⊂ f(B(x)) for all x ∈ X; (iii) If a p-stack F ω-converges to x, then the image p-stack f(F) ω-converges to f(x).

Proof. (i) ⇒ (ii): Let V be any member of B(f(x)) in Y. Then there is a ω-open set W such that W ⊂ V. Since f is ω-irresolute, there exists a ω-open neighborhood U ∈ B(x) such that f(U) ⊂ W ⊂ V , thus V ∈ f(B(x)). (ii) ⇒ (iii): It is obvious. (iii) ⇒ (i): If f is not ω-irresolute, then for some x ∈ X, there is a ω-open neighborhood V ∈ B(f(x)) such that for all ω-open neighborhood U ∈ B(x), f(U) is not included in V. For all U ∈ B(x), since f(U)∩(Y\V)⊭ ∅, we get a p-stack F = f(B(x))∪<Y\V >. And since U ∩f−1(Y\V)⊭ ∅, also we get a p-stack G = B(x)∪f−1<Y\V > which ω-converges to x. But since f(G) is a finer p-stack than F and Y\V ∈ F, f(G) can't ω-converge to f(x), contradicting to (iii).

Now we introduce a new notion of p-ω-compactness by p-stacks and investigate the related properties.

Definition 3.12. Let (X,τ) be a topological space and A be a subset of X. A subset A of a topological space (X,τ) is p-ω-compact if every ultrapstack containing A ω-converges to a point in A. A topological space (X,τ) is p-ω-compact if X is p-ω-compact.

Let X = {a,b,c}. In case τ is the discrete topology, let H be an ultrapstack containing a p-stack F generated by {{a,b},{b,c},{a,c}}. Then it doesn't ω-converge to any point in X. Thus the topological space (X,τ) is not p-ω-compact. But in case τ = {∅,{a},{b,c},X}, the topological space (X,τ) is p-ω-compact.

Theorem 3.13. If a topological space (X,τ) is p-ω-compact and A ⊂ X is ω-closed, then A is p-ω-compact.

Proof. Let F be an ultrapstack containing A. From Definition 3.12, there is x ∈ X such that F ω-converges to x. Thus B(x) ⊂ F, and since A ∈ F and F is a p-stack, A∩V⊭ ∅ for all V ∈ B(x). So by

Theorem 3.4, we can say x ∈ ω Cl(A) = A.

Theorem 3.14. The ω-irresolute image of a p-ω-compact set is p-ω-compact.

Proof. Let a function f : (X,τ) → (Y,σ) be ω-irresolute, let AX be p-ω-compact, and let H be an ultrapstack containing f(A). If G is an ultrapstack containing the p-stack base {f−1(H) : H ∈ H}∪<A>, then for some x ∈ A, G ω-converges to x, and H = f(G) ω-converges to f(x). Thus, f(A) is p-ω-compact.

Theorem 3.15. A topological space (X,τ) is p-ω-compact if and only if each ω-open cover of X has a two-element subcover.

Proof. Suppose H is an ultrapstack in X such that it doesn't ω-converge to any point in X. Then for each x ∈ X, there is a ω-open subset Ux ∈ B(x) such that Ux / ∈ H. By Lemma 2.4(iii), X\Ux ∈ H, for allx ∈ X. Thus U = {Ux : x ∈ X} is a ω-open cover of X. But U has no two-element subcover of X, for if U,V ∈ U and X ⊂ U ∪V , then (X\U)∩(X\V) = X\(U ∪V) = ∅, contradicting the assumption that H is a p-stack. Conversely, let U be a ω-open cover of X with no two-element subcover of X. Then B = {X\U: U ∈ U} is p-stack base, and any ultrapstack containing B cannot ω-converge to any point in X.

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